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An Insight into the Solitary Wave Solutions of the 2D-NLSE: Existence Conditions and Parameter Dependence

[1] Hanan A Alkhidhr, [2] Aaliyah TS Alharbi

[1, 2] Department of Mathematics, College of Science, Qassim University, Buraidah, Saudi Arabia

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Corresponding Author: **Hanan A Alkhidhr**

Abstract

This study derives and examines a bright solitary-wave solution of the two-dimensional cubic nonlinear Schrödinger equation (2D-NLSE) by combining He's semi-inverse variational method with the Ritz approach. A complex traveling-wave transformation is used to convert this equation to ordinary differential equation, whereas the decomposition of the real and imaginary components produces an explicit compatibility requirement governing the propagation velocity. A variational functional is created for the reduced equation, and a hyperbolic-secant solution is used as the Ritz trial function. By performing direct substitution, one obtains an independent verification of exactness while simultaneously establishing the amplitude and inverse-width requirements. The analysis indicates that the effective frequency and cubic nonlinearity dictate the maximum amplitude, whereas the effective frequency and directional dispersion determine the pulse width. Furthermore, the simulation presentations for the proposed solutions were displayed using MATLAB software.

Keywords: Solitary Waves, 2D-NLSE, Physical Coefficients, Variational Principle, Physical Applications

1. Introduction

Recently, a lot of attention has been paid to nonlinear partial differential equations (NPDEs) that arise in numerous disciplines of mathematical and physical sciences, such as plasma physics, economy, fluid dynamics, quantum mechanics, optical fibers, and many others [1, 2, 3]. These equations are usually used to describe many complex nonlinear phenomena. In this broad category, the nonlinear Schrödinger equation (NLSE) is a crucial model for the rivalry between dispersive spreading and nonlinear self-phase modulation [4, 5]. This equation plays an important role in the mathematical modelling of wave phenomena in fluid dynamics, optics and quantum systems. Bright solitary waves are localized wave packets that form and propagate as solutions to the NLSE. These solutions serve as analytical building blocks for both numerical simulations and idealised physical states.

The systematic construction of traveling-wave solutions for NPDEs is based on reducing a complex NPDE into a simpler ordinary differential equation (ODE) using a proper traveling-wave transformation [6, 7, 8]. Many efficient and potent techniques have been put forth thus far to get exact solutions of nonlinear partial differential equations, including He's semi-invers method, exp-function method, F-expansion method, Hirota's bilinear method, tanh-function method, Darboux transformations, Jacobian elliptic equation expansion method and so on. Because the geometric and mathematical terrain of nonlinearity is so complex, no single approach works for all nonlinear models [9, 10, 11].

Nonlinear waves are becoming more common in mathematical physics and engineering fields. These waves constitute essential components across numerous scientific and technological domain. The comprehensive mathematical examination of nonlinear wave dynamics through NPDEs provides critical understanding of the underlying mechanisms governing intricate wave phenomena and facilitates the determination of precise solutions that elucidate the sophisticated interactions among dispersion, nonlinearity, and additional physical effects [12, 13]. Thus, rigorous analysis of nonlinear waves is critical for improving theoretical frameworks, anticipating complicated dynamical behaviours, and driving novel applications in modern science and engineering. Because of their ubiquity, the study of nonlinear wave propagation in dispersive media has received significant interest.

In two spatial dimensions, the dispersion coefficients in the x- and y-directions need not be equal. This anisotropy is represented here by separate coefficients, and the resulting effective dispersion depends on the orientation of the traveling coordinate. Consequently, the existence and width of a localized profile are linked not only to the nonlinear coefficient but also to the propagation direction and carrier phase. The present work revisits the application of He's semi-inverse method and the

Ritz variational approach [14, 15, 16], to the two-dimensional cubic NLSE. Although the solitary wave solutions of reduced forms of the NLSE are well established, a transparent derivation that connects a variational construction, a Ritz trial profile, and direct coefficient matching remains useful: the variational step provides a systematic approximation framework, while substitution into the reduced equation determines whether the selected profile is exact. This study provides a consistent derivation of the bright sech solution, an explicit statement of the speed compatibility and existence criteria, and a parameter-based interpretation of amplitude and width. Additionally, Matlab program was used to display the simulation presentations for the gained solutions.

This paper is constructed as follows. Section 2 introduces the governing equation and the corresponding mathematical symbols. Section 3 presents the traveling-wave transformation and deduces the compatibility condition. Section 4 gives variational formulation via He's semi-inverse technique. Section 5 uses the Ritz methodology to create the bright solitary-wave solution and establish its admissibilityconditions. Section 6 discusses the Ritz stationarity conditions in relation to exactness. Section 7 investigates the parameter dependency of the solution and explores the physical applications. Section 8 concludes with a remark regarding the results obtained and future work.

2. Governing equation and notation

We consider the two-dimensional cubic nonlinear Schrödinger equation:

$$iq_t + \alpha q_{xx} + \beta q_{yy} + \gamma |q|^2 q + \delta q = 0 \tag{2.1}$$

The complex-valued field q(x,y,t) represents the wave amplitude; whereas x and y denote spatial coordinates and t is time. The real constants α and β symbolize the dispersion coefficients in the x- and y-directions; γ denotes the cubic nonlinear coefficient; and δ is a constant linear potential. The carrier wavenumbers are k and ℓ, and ω is the carrier frequency. The notation outlined in Table 1 is utilized consistently across the document.

Table 1: Definition of mathematical symbols and model parameters

Symbol	Meaning
q(x,y,t)	Complex wave field
x, y, t	Spatial coordinates and time
α, β	Dispersion coefficients along x and y
γ	Cubic nonlinear coefficient
δ	Constant linear potential / frequency shift
k, ℓ, ω	Carrier wavenumbers and carrier frequency
a, b, c	Traveling-coordinate scales and envelope speed
$\xi = ax + by - ct$	Traveling coordinate
u(ξ)	Real-valued envelope
$D = \alpha a^2 + \beta b^2$	Effective directional dispersion
$\Omega = \omega - \alpha k^2 - \beta \ell^2 + \delta$	Effective linear-frequency parameter
A, B	Sech-profile amplitude and inverse width
W = 1/B	Characteristic width of the envelope
J[u]	Action functional associated with the reduced equation

3. Traveling-wave reduction

Introduce the traveling-wave form:

$$q(x,y,t) = u(\xi)\exp[i(kx + \ell y - \omega t)], \quad \xi = ax + by - ct \tag{3.1}$$

Here u(ξ) is a real-valued envelope, ξ is the traveling coordinate, a and b set its directional scaling, and c is the envelope propagation speed. The parameters k and ℓ are the carrier wavenumbers along x and y, and ω is the carrier frequency. Substituting Eq. (3.1) into Eq. (2.1) and separating the real and imaginary parts. The imaginary part gives the following compatibility condition:

$$-c + 2\alpha ak + 2\beta b\ell = 0 \tag{3.2}$$

Thus, the traveling-wave parameter must satisfy $c = 2\alpha ak + 2\beta b\ell$. The real part gives:

$$D\,u'' + \Omega u + \gamma u^3 = 0, \tag{3.3}$$

Where the reduced coefficients are defined by:

$$D = \alpha a^2 + \beta b^2, \ \Omega = \omega - \alpha k^2 - \beta \ell^2 + \delta \tag{3.4}$$

The quantity D is the effective dispersion along the chosen traveling direction, while Ω is the effective linear-frequency parameter after accounting for the carrier wavenumbers and the constant shift δ. We assume $D \neq 0$ in the reduction. These definitions are retained throughout the paper; no alternative symbols are introduced for the same physical quantities.

4. He's semi-inverse variational formulation

For the reduced equation $Du'' + \Omega u + \gamma u^3 = 0$, a suitable Lagrangian is:

$$L(u,u') = \frac{D}{2}(u')^2 - \frac{\Omega}{2}u^2 - \frac{\gamma}{4}u^4 \tag{4.1}$$

The associated action is $J[u] = \int L(u,u')\, d\xi$. With the convention $\frac{\partial L}{\partial u} - \frac{d}{d\xi}\left(\frac{\partial L}{\partial u'}\right) = 0$, the Euler–Lagrange equation becomes $-\Omega u - \gamma u^3 - D u'' = 0$, which is equivalent to (3.3) after multiplication by -1. Thus, the reduced equation has the variational structure required for the semi-inverse procedure. For a localized solution, the action integral is interpreted over the real line, with u and its derivative decaying sufficiently rapidly for the relevant integrals to converge.

He's semi-inverse method is used here to identify a suitable variational structure for the reduced equation. The Ritz method then restricts the admissible profiles to a parameterized family and determines stationary values of the reduced action. Because a stationary Ritz profile is not automatically an exact solution, the candidate obtained below is also substituted directly into the governing ordinary differential equation.

5. Ritz approximation and exact sech solution

Choose the hyperbolic-secant trial function:

$$u(\xi) = A\,\mathrm{sech}(B\xi),\ A > 0,\ B > 0. \tag{5.1}$$

The amplitude A is taken positive, and $B > 0$ is the inverse-width parameter; the characteristic width is therefore proportional to $\frac{1}{B}$. The sech profile is localized because it decays exponentially as $|\xi|$ grows.

$$\frac{d^2}{d\xi^2}\mathrm{sech}(B\xi) = B^2\,\mathrm{sech}(B\xi) - 2B^2\,\mathrm{sech}^3(B\xi) \tag{5.2}$$

Allows direct verification of the trial profile in (3.3). Substitution gives:

$$(D B^2 + \Omega) A\,\mathrm{sech}(B\xi) + (\gamma A^3 - 2DB^2 A)\,\mathrm{sech}^3(B\xi) = 0. \tag{5.3}$$

Since $\mathrm{sech}(B\xi)$ and $\mathrm{sech}^3(B\xi)$ are linearly independent functions, both coefficients must vanish. Hence:

$$DB^2 + \Omega = 0,\ \gamma A^2 - 2DB^2 = 0. \tag{5.4}$$

The parameters are therefore:

$$B^2 = -\frac{\Omega}{D},\quad A^2 = -\frac{2\Omega}{\gamma} \tag{5.5}$$

The bright sech wave is real and localized when $\frac{\Omega}{D} < 0$ and $\frac{\Omega}{\gamma} < 0$. Under these conditions, the solution is:

$$q(x,y,t) = \sqrt{-\frac{2\Omega}{\gamma}}\ \mathrm{sech}\left(\sqrt{-\frac{\Omega}{D}}\,(ax + by - ct)\right)\exp[i(kx + \ell y - \omega t)] \tag{5.6}$$

The bright sech wave is admissible provided $D \neq 0$ and both radicands are strictly positive, namely $\frac{\Omega}{D} < 0$ and $\frac{\Omega}{\gamma} < 0$. These inequalities require Ω to have the opposite sign to both D and γ. If either radicand is zero or negative, this real, localized bright-sech solution is not defined by (5.6); other solution types or degenerate limits would need separate analysis.

6. Ritz stationarity conditions

In the Ritz implementation, the trial function (5.1) is inserted into the action functional and the resulting reduced functional $J(A,B)$ is made stationary with respect to the free parameters. The stationary equations are:

$$\frac{\partial J}{\partial A} = 0,\quad \frac{\partial J}{\partial B} = 0$$

The Ritz equations provide variational estimates within the chosen trial family. For this particular cubic equation, however, direct substitution yields two independent powers of sech, so coefficient matching gives the conditions (5.4) exactly. Accordingly, the profile in (5.6) is an exact solution when those conditions and the positivity constraints are satisfied, rather

than merely a Ritz approximation. The Ritz technique should be reported as a variational construction, with direct replacement used as an exactness test.

7. Discussion and parameter dependence

The solution parameters expose a clear separation of roles. The peak envelope amplitude and inverse width are $A = \sqrt{-\frac{2\Omega}{\gamma}}$ and $B = \sqrt{-\frac{\Omega}{D}}$, respectively, while the characteristic width is:

$$W = \frac{1}{B} = \sqrt{-\frac{D}{\Omega}}$$

Interpreted only in the admissible regime. Thus, Ω influences both amplitude and width; γ directly controls the amplitude, whereas D directly controls the width. In particular, increasing $|\gamma|$while holding Ω fixed reduces A, whereas increasing $|\Omega|$ within a fixed admissible sign regime increases A. The dependence on D does not alter the peak amplitude in this reduced formula, but it changes the spatial scale of the pulse.
These trends can be expressed through local relative sensitivities, away from the degenerate limits $\Omega = 0$, $D = 0$, and $\gamma = 0$:

$$\frac{dA}{A} = \frac{1}{2}\left(\frac{d\Omega}{\Omega} - \frac{d\gamma}{\gamma}\right)$$

$$\frac{dB}{B} = \frac{1}{2}\left(\frac{d\Omega}{\Omega} - \frac{dD}{D}\right)$$

And

$$\frac{dW}{W} = \frac{1}{2}\left(\frac{dD}{D} - \frac{d\Omega}{\Omega}\right)$$

These differential relations describe small parameter changes while the solution remains admissible; they should not be interpreted as permitting arbitrary sign changes across an existence boundary. The effective dispersion $D = \alpha a^2 + \beta b^2$ combines the directional dispersion coefficients with the orientation and scaling of the traveling coordinate. Hence, changes in α, β, a, or b can alter the width through D. The effective parameter $\Omega = \omega - \alpha k^2 - \beta \ell^2 + \delta$ combines the carrier frequency, carrier wavenumbers, dispersion, and linear shift; changes in these quantities can therefore affect both amplitude and width through Ω. The speed condition $c = 2\alpha a k + 2\beta b \ell$ couples the envelope speed to the carrier wavenumbers and the directional dispersion. The constants a and b also determine the orientation of ξ, so the pulse is localized across the traveling coordinate rather than representing a radially symmetric two-dimensional lump.
Fig 1 illustrates how the solitary-wave profile responds to changes in α. With the other parameters held fixed, α enters both the effective directional dispersion $D = \alpha a^2 + \beta b^2$ and the effective frequency $\Omega = \omega - \alpha k^2 - \beta \ell^2 + \delta$. Thus, changing α may alter the pulse width through D and may also alter its peak amplitude and width through Ω. The direction and magnitude of these changes cannot be stated universally without the parameter values because the two effects may reinforce or counteract one another. The comparison is valid as a bright-sech solution only for parameter choices satisfying $\frac{\Omega}{D} < 0$ and $\frac{\Omega}{\gamma} < 0$.
Fig 2 highlights the effect of the cubic nonlinear coefficient γ on the solitary-wave profile. For fixed Ω and D, the amplitude satisfies $A = \sqrt{-\frac{2\Omega}{\gamma}}$, so its magnitude scales as $|\gamma|^{-\frac{1}{2}}$. A larger $|\gamma|$ produces a lower peak, while a smaller $|\gamma|$ produces a higher peak, provided the sign conditions remain admissible. Because $B = \sqrt{-\frac{\Omega}{D}}$ does not contain γ, the inverse width and characteristic width remain unchanged when only γ is varied. Accordingly, the curves in Fig 2 should differ in peak height while retaining the same spatial scale, assuming all other parameters are identical. Only values of γ for which $\frac{\Omega}{\gamma} < 0$ give the real bright-sech profile considered here; a sign change that violates this condition takes the solution outside the stated existence regime.

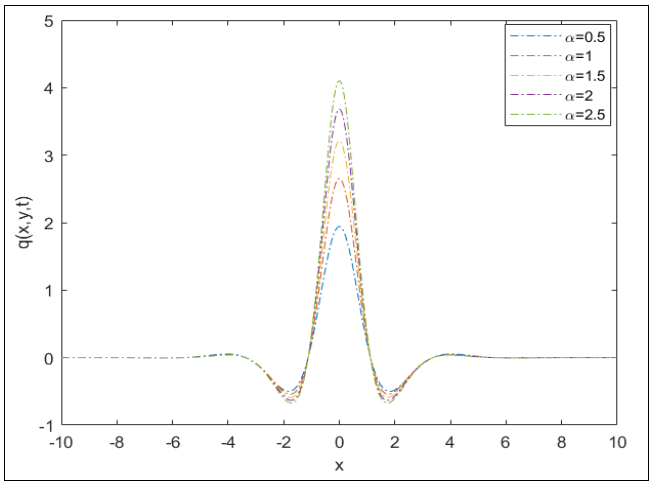


Fig 1: Effect of the dispersion coefficient α on the traveling-wave solution (5.6)

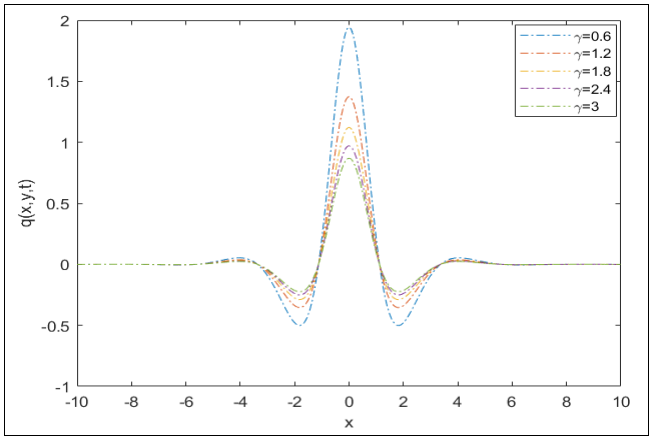


Fig 2: Effect of the cubic nonlinear coefficient γ on the traveling-wave solution (5.6)

8. Conclusions

This paper provided an analytical framework to examine solitary-wave solutions of the two-dimensional cubic nonlinear Schrödinger problem by integrating the semi-inverse variational method with the Ritz approach. The traveling-wave reduction established the propagation-speed compatibility requirement and the accompanying variational formulation, whilst direct replacement validated the hyperbolic-secant solution's accuracy under the required existence conditions. The investigation showed that while the cubic nonlinear coefficient controls the amplitude and the effective directional dispersion controls the characteristic width, the effective frequency affects both the wave's amplitude and width. The impact of dispersion, carrier characteristics, and propagation direction on the final wave profile was further elucidated by the parameter research. Instead of a fully localised two-dimensional structure, the derived solution depicts a stripe-like solitary wave because the envelope depends on a single travelling coordinate.

9. References

1. Malik HK, Srivastava R, Kumar S, Singh D. Small amplitude dust acoustic solitary wave in magnetized two ion temperature plasma. Journal of Taibah University for Science. 2020; 14(1):417-422.
2. Younis M, Ali S, Mahmood SA. Solitons for compound KdV Burgers equation with variable coefficients and power law nonlinearity, Nonlinear Dyn. 2015; 81:1191-1196.
3. Ghanbari B, Kuo C. New exact wave solutions of the variable coefficient (1 + 1)-dimensional Benjamin-Bona-Mahony and (2 + 1)-dimensional asymmetric NizhnikNovikov-Veselov equations via the generalized exponential rational function method. Eur. Phys. J. Plus. 2019; 134:334.
4. Khan A, Muhammad J, Younas U, Thinakaran R, Abdeljawad T, Alqudah MA. Investigating the stochastic higher dimensional nonlinear Schrodinger equation to telecommunication engineering, Scientific Reports. 2025; 15:27309.
5. Liu YH, Guo R, Li XL. Rogue wave solutions and modulation instability for the mixed nonlinear Schrodinger equation. Appl. Math. Lett. 2021; 121:107450.
6. Chen YQ, Tang YH, Manafian J, Rezazadeh H, Osman MS. Dark wave, rogue wave and perturbation solutions of Ivancevic option pricing model, Nonlinear Dyn. 2021; 105:2539-2548.
7. Matinfar M, Hosseini K. Optical solitons of (2+1)-dimensional nonlinear Schrodinger equation involving linear and nonlinear effects, Optik. 2021; 228:166110.
8. Pandir Y, Ekin A. New solitary wave solutions of the Korteweg-de Vries (KdV) equation by new version of the trial equation method, Electron J Appl Math. 2023; 1(1):101-113.

9. Rehman HU, Imran MA, Ullah N, Akgul A. Exact solutions of (2+1)-dimensional Schr¨odinger's hyperbolic equation using different techniques, Numer Methods Partial Differential Eq., 2020, 1-20.
10. Sultan AM, Lu D, Arshad M, Rehman HU, Saleem MS. Soliton solutions of higher order dispersive cubic-quintic nonlinear Schrodinger equation and its applications. Chin. J. Phys. 2020; 67:405-413.
11. Liu MZ, Cao XQ, Zhu XQ, Liu BN, Peng KC. Variational principles and solitary wave solutions of generalized nonlinear Schrodinger equation in the ocean. Journal of Applied and Computational Mechanics. 2021; 7(3):1639-1648.
12. Ma YL. Interaction and energy transition between the breather and rogue wave for a generalized nonlinear Schrodinger system with two higher-order dispersion operators in optical fibers, Nonlinear Dynamics. 2019; 97:95-105.
13. Gao XY, Guo YJ, Shan WR. Optical waves/modes in a multicomponent inhomogeneous optical fiber via a three-coupled variable-coefficient nonlinear Schrodinger system, Appl. Math. Lett. 2021; 120:107161.
14. He JH. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. Int. J. Turbo JetEngines. 1997; 14(1):23-28.
15. He JH. Variational principles for some nonlinear partial differential equations with variable coencients. Chaos Solitons Fractals. 2004; 19(4):847-851.
16. He JH. Some asymptotic methods for strongly nonlinear equations, Internat, J. Modern Phys. B. 2006; 20:1141-1199.