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A Malcev-Type Identity for Binary Leibniz Algebras

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Abstract
Let $D(a,b,c) = (ab)c - a(bc) + b(ac)$ be the Leibniz defect over a field of characteristic zero. We study the subvariety of binary Leibniz algebras defined by the single additional identity $D(a,b,ac) = D(a,b,c)a$. It contains both Leibniz and Malcev algebras and is a proper subvariety of the binary Leibniz variety. This gives an affirmative answer to the question posed by Dzhumadil'daev and Ismailov about a Leibniz analogue of Malcev algebras lying strictly between the Leibniz and binary Leibniz varieties. We prove that its additional multilinear relations in degree four form the irreducible S_4-module $S^{(2,1,1)}$. The proof uses three elementary symmetries and an explicit evaluation certificate. A known four-dimensional family provides concrete examples and counterexamples, with the identity verified for arbitrary elements.

Keywords: Binary Leibniz Algebra, Malcev Algebra, Polynomial Identity, Subvariety, Specht Module

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1. Introduction

Leibniz and Malcev algebras extend Lie algebras in different directions. The Leibniz identity retains the derivation form of the Jacobi identity without requiring anticommutativity. The Malcev identity weakens the Jacobi identity within the class of anticommutative algebras. Both classes belong to the variety of binary Leibniz algebras, whose two-generated subalgebras are Leibniz algebras.

Ismailov and Dzhumadil'daev gave a finite system of defining identities for binary Leibniz algebras [2]. Their characterization makes it possible to study intermediate varieties by imposing further identities on the Leibniz defect. A natural question is whether one quartic identity can define a proper subvariety which still contains every Leibniz algebra and every Malcev algebra.

In this note we consider:

$$D(a,b,ac) = D(a,b,c)a \tag{1.1}$$

This is automatic when $D = 0$, and in an anticommutative algebra it is precisely the Malcev identity. The main point is to determine the multilinear relation module contributed by (1.1) inside the binary Leibniz variety. We show that it is three-dimensional and irreducible, of type $(2,1,1)$.

The argument is explicit. We derive symmetries of the linearized identity, obtain the action of the adjacent transpositions, and prove independence by three evaluations in a four-dimensional algebra. The test family is classical; see [1, Section 2]. Its role here is to provide a short verification of the chosen identity and a concrete certificate for the representation-theoretic argument.

2. The identity and its elementary symmetries

All algebras are nonunital algebras over a field $\mathbb{F}$ of characteristic zero, with one bilinear multiplication. Put:

$$D(a,b,c) = (ab)c - a(bc) + b(ac)$$

We use the left Leibniz convention $D(a,b,c) = 0$ and denote the corresponding variety by $Leib$. By the characterization of Ismailov and Dzhumadil'daev [2, Theorem 1.2 of the preprint], the variety $\mathcal{B}$ of binary Leibniz algebras is defined by:

$$D(a,a,b)=0, \qquad D(a,b,a)=0, \qquad D(a,b,ab)=0 \tag{2.1}$$

Polarizing the first two identities gives:

$$D(a,b,c)=-D(b,a,c)=-D(c,b,a) \tag{2.2}$$

Thus D is alternating in all three arguments in every algebra of $\mathcal{B}$. In particular, $(ab)c+(ba)c=0$.
Set:

$$m(a,b,c)=D(a,b,ac)-D(a,b,c)a, \qquad \mathcal{C}=\{A\in\mathcal{B}\mid m=0\} \tag{2.3}$$

Linearization in the repeated variable gives the multilinear polynomial:

$$\begin{aligned} g(a,b,c,d)= \quad & D(a,b,cd)+D(c,b,ad) \\ & -D(a,b,d)c-D(c,b,d)a \end{aligned} \tag{2.4}$$

Indeed,

$$g(a,b,c,d)=m(a+c,b,d)-m(a,b,d)-m(c,b,d), \quad g(a,b,a,d)=2m(a,b,d)$$

Consequently $m=0$ and $g=0$ are equivalent identities.
Lemma 2.1: *Modulo the identities of $\mathcal{B}$, the polynomial g satisfies:*

$$g(a,b,c,d)=g(c,b,a,d) \tag{2.5}$$

$$g(a,b,c,d)=-g(a,d,c,b) \tag{2.6}$$

$$g(a,b,c,d)+g(b,a,c,d)+g(a,c,b,d)=0 \tag{2.7}$$

Proof. The first equality follows directly from (2.4). The complete linearization of $D(a,b,ab)=0$ is:

$$D(a,b,cd)+D(c,b,ad)+D(a,d,cb)+D(c,d,ab)=0 \tag{2.8}$$

It is obtained by taking the component involving each of a,b,c,d once in $D\big(a+c,b+d,(a+c)(b+d)\big)$. On adding $g(a,b,c,d)$ and $g(a,d,c,b)$, the terms ending in a or c cancel by alternation of D. The remaining terms are exactly the left side of (2.8), proving (2.6). Finally, expand the three summands in (2.7) using (2.4). The terms $D(a,b,cd)$ and $D(b,a,cd)$ cancel, as do the analogous terms with final arguments ad and bd. The six remaining terms cancel in pairs by (2.2).

3. A four-dimensional test family

For $\lambda\in\mathbb{F}$, let A_λ be the anticommutative algebra with basis e_1,e_2,e_3,e_4 and nonzero products, up to anticommutativity,

$$e_1e_2=e_2, \quad e_1e_3=e_3, \quad e_2e_3=e_4, \quad e_1e_4=\lambda e_4 \tag{3.1}$$

Unlisted products are zero. This is the familiar one-parameter family of four-dimensional binary Lie algebras appearing in [1, Section 2], with a relabeling of the basis. We include the calculations needed below.
Proposition 3.1: *Every A_λ is binary Leibniz. Moreover,*

$$A_\lambda\in\mathcal{C} \quad\Leftrightarrow\quad \lambda\in\{-1,2\}$$

The algebra A_2 is Lie, whereas A_{-1} is Malcev and is not Leibniz. In particular, A_0 is binary Leibniz but does not belong to $\mathcal{C}$.
Proof. In an anticommutative algebra the defect equals the Jacobian:

$$D(x,y,z)=J(x,y,z)=(xy)z+(yz)x+(zx)y$$

For $x=\sum_i x_i e_i$, $y=\sum_i y_i e_i$, and $z=\sum_i z_i e_i$, write:

$$\Delta(x,y,z)=\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

The Jacobian is alternating and trilinear. The multiplication table gives:

$$J(e_1, e_2, e_3) = e_4 - \lambda e_4 + e_4 = (2 - \lambda)e_4$$

The Jacobians on the other three increasing triples are zero: each term of $J(e_1, e_2, e_4)$, $J(e_1, e_3, e_4)$, and $J(e_2, e_3, e_4)$ vanishes by (3.1). Therefore:

$$D(x, y, z) = (2 - \lambda)\Delta(x, y, z)e_4 \tag{3.2}$$

This basis check is legitimate because the Jacobian is multilinear.
To verify the identities with repeated variables, we now use arbitrary elements. The product is:

$$\begin{aligned} xy = \quad & (x_1y_2 - x_2y_1)e_2 + (x_1y_3 - x_3y_1)e_3 \\ & + \big(x_2y_3 - x_3y_2 + \lambda(x_1y_4 - x_4y_1)\big)e_4 \end{aligned} \tag{3.3}$$

If $\pi(x) = (x_1, x_2, x_3)$, then:

$$\pi(xy) = x_1\pi(y) - y_1\pi(x)$$

It follows that $\Delta(x, y, xy) = 0$, and hence $D(x, y, xy) = 0$. The other two identities in (2.1) follow from alternation. Thus $A_\lambda \in \mathcal{B}$ for every λ.
Similarly, $\pi(xz) = x_1\pi(z) - z_1\pi(x)$ gives:

$$D(x, y, xz) = (2 - \lambda)x_1\Delta(x, y, z)e_4$$

Since $e_4x = -\lambda x_1 e_4$, we also have:

$$D(x, y, z)x = -\lambda(2 - \lambda)x_1\Delta(x, y, z)e_4$$

Subtracting yields the complete evaluation formula:

$$m(x, y, z) = (2 - \lambda)(1 + \lambda)x_1\Delta(x, y, z)e_4 \tag{3.4}$$

It vanishes identically exactly when $\lambda = 2$ or $\lambda = -1$: necessity follows by taking $(x, y, z) = (e_1, e_2, e_3)$. Equation (3.2) shows that A_2 is Lie and that $D(e_1, e_2, e_3) = 3e_4 \neq 0$ in A_{-1}. In the anticommutative case $m = 0$ is the Malcev identity, so A_{-1} is Malcev. Finally, in A_0 one has $m(e_1, e_2, e_3) = 2e_4 \neq 0$.

4. The subvariety and its quartic relation module

Let P_4 be the multilinear component in a, b, c, d of the absolutely free nonassociative algebra. For a variety $\mathcal{U}$, write $P_4(\mathcal{U}) = P_4/(P_4 \cap \mathrm{Id}(\mathcal{U}))$ and $c_4(\mathcal{U}) = \dim P_4(\mathcal{U})$. The symmetric group S_4 acts by permuting the variable labels; for example, $(12)g = g(b, a, c, d)$.

Theorem 4.1: *The variety $\mathcal{C}$ defined in (2.3) satisfies:*

$$Leib \subsetneq \mathcal{C} \subsetneq \mathcal{B}, \qquad Malc \subsetneq \mathcal{C}$$

Its anticommutative members are exactly the Malcev algebras. Furthermore, its additional multilinear relation space in degree four is:

$$\frac{P_4 \cap \mathrm{Id}(\mathcal{C})}{P_4 \cap \mathrm{Id}(\mathcal{B})} = \mathrm{span}_{\mathbb{F}}\{g, (12)g, (14)g\} \cong S^{(2,1,1)} \tag{4.1}$$

In particular, $c_4(\mathcal{C}) = c_4(\mathcal{B}) - 3$.

Proof. Every Leibniz algebra satisfies $m = 0$, since $D = 0$. If an algebra is anticommutative, then $D = J$ and $m = 0$ becomes $J(a, b, ac) = J(a, b, c)a$, the Malcev identity. A Malcev algebra satisfies $J(a, b, ab) = 0$ by setting $c = b$; thus it is binary Leibniz by (2.1). This proves the inclusions and the assertion about anticommutative members. Proposition 3.1 gives $A_{-1} \in \mathcal{C} \setminus Leib$ and $A_0 \in \mathcal{B} \setminus \mathcal{C}$. The two-dimensional algebra with basis u, v and sole nonzero product $u^2 = v$ is Leibniz, because every triple product is zero. It belongs to $\mathcal{C}$ but is not anticommutative, proving $Malc \subsetneq \mathcal{C}$.
It remains to prove (4.1). Work modulo $\mathrm{Id}(\mathcal{B})$ and set:

$$g_1 = g, \qquad g_2 = (12)g, \qquad g_3 = (14)g$$

The three symmetries in Lemma 2.1 give the following action table:

	g_1	g_2	g_3
(12)	g_2	g_1	$-g_3$
(23)	$-g_1-g_2$	g_2	g_2-g_3
(34)	$-g_1-g_3$	$-g_2+g_3$	g_3

(4.2)

For clarity, the first entry of the second row is (2.7). Applying that same identity to (a,d,c,b) and using (2.6) gives:

$$-g(a,b,c,d)+g(d,a,c,b)-g(a,b,d,c)=0$$

Here $g(d,a,c,b)=-g(d,b,c,a)=-g_3$, giving the first entry of the last row. The remaining entries follow by the same variable permutations and (2.5)–(2.6). Since adjacent transpositions generate S_4, the table proves that the orbit of g is contained in $\mathrm{span}\{g_1,g_2,g_3\}$.

To prove independence, evaluate in A_0. Polarizing (3.4) gives:

$$g(a,b,c,d)=2\big(a_1\Delta(c,b,d)+c_1\Delta(a,b,d)\big)e_4 \tag{4.3}$$

The coefficients of e_4 in the three evaluations below are:

(a,b,c,d)	g_1	g_2	g_3
(e_1,e_2,e_1,e_3)	4	−2	−2
(e_1,e_1,e_2,e_3)	−2	−2	0
(e_1,e_2,e_3,e_1)	−2	0	−2

$$\det\begin{pmatrix}4&-2&-2\\-2&-2&0\\-2&0&-2\end{pmatrix}=32$$

(4.4)

Since $A_0\in\mathcal{B}$, any linear dependence modulo $\mathrm{Id}(\mathcal{B})$ would vanish on these substitutions. The nonzero determinant excludes such a dependence. Thus g_1,g_2,g_3 form a basis of the orbit module.

Using (4.2), its character on the conjugacy classes $1,(12),(12)(34),(123),(1234)$ is:

$$(3,-1,-1,0,1)$$

The class sizes are $1,6,3,8,6$, and the character has self-inner product $(9+6+3+0+6)/24=1$. It is therefore irreducible. It is the sign twist of the standard representation of S_4, namely $S^{(2,1,1)}$; see [3].

Finally, $\mathcal{C}$ is defined over $\mathcal{B}$ by the homogeneous quartic identity $m=0$, equivalently by $g=0$. In a free nonunital algebra, multiplication by a nonconstant word or substitution of a product increases degree. Hence the degree-four multilinear consequences of g are precisely its S_4-span: linear substitutions and multilinear extraction give only linear combinations of variable permutations. This identifies that span with the left side of (4.1) and proves the dimension formula.

For a concrete member of $\mathcal{C}$ which is neither Leibniz nor Malcev, take $A_{-1}\oplus\mathrm{span}\{u,v\}$ with $u^2=v$ and all mixed products zero. Polynomial identities hold componentwise, so this six-dimensional algebra belongs to $\mathcal{C}$. Its nonzero square excludes anticommutativity, while $D(e_1,e_2,e_3)=3e_4$ excludes the Leibniz identity. This is the direct-sum construction of [2, Example 1.5 of the preprint], now viewed inside $\mathcal{C}$.

The theorem isolates the effect of one additional identity. It does not assert that $\mathcal{C}$ is the least variety containing $Leib$ and $Malc$, nor does it determine their common identities in higher degrees. Those questions require information beyond the single quartic module considered here.

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