Int. j. adv. multidisc. res. stud. 2026; 6(5):1336-1341

Received: 25-08-2026
Accepted: 05-10-2026

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

Reciprocal Irregularity Under Partial Edge Subdivision

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DOI: https://doi.org/10.62225/2583049X.2026.6.5.7021
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Abstract
Edge subdivision is a basic graph transformation, but different degree-irregularity measures need not react to it in the same way. This paper determines the exact response of reciprocal irregularity when an arbitrary set of edges is subdivided, with each selected edge allowed to receive any positive number of new degree-two vertices. A local formula is obtained from the endpoint degrees, yielding an additive law for partial subdivision, a complete equality characterization, invariance with respect to repeated subdivision along the same original edge, and an immediate fixed-budget edge-selection principle. Reciprocal irregularity is shown never to decrease; outside the exceptional graph K2, a selected set produces a strict increase precisely when it contains an edge whose two original endpoints both have degree at least three. A comparative theorem proves that the Albertson-irregularity change equals 2 min{d(u), d(v)} times the reciprocal change, so the two measures have identical local sign behavior. In contrast, the sigma irregularity changes by 2(d(u)-2)(d(v)-2) and therefore decreases on pendant edges incident with a vertex of degree at least three. Extensive exact-arithmetic Graph Atlas verification found no counterexample to the stated identities and classifications.

Keywords: Reciprocal Irregularity, Partial Edge Subdivision, Albertson Irregularity, Sigma Index, Graph Operations

1. Introduction

Graph irregularity can be quantified in several nonequivalent ways. Albertson's classical irregularity is the sum, over all edges, of the absolute degree difference of the two endpoints [1]. More recently, Chen and Liu introduced the reciprocal irregularity:

$$\mathrm{irr}_r(G) = \sum_{uv \in E(G)} \left| \frac{1}{d_G(u)} - \frac{1}{d_G(v)} \right|$$

And studied its extremal behavior for connected graphs, bipartite graphs and trees [2]. Their results show that replacing degrees by reciprocal degrees changes the extremal behavior substantially, even though both indices vanish exactly on regular connected graphs. More broadly, established graph-irregularity measures need not rank graphs consistently, which makes comparison under a common graph transformation mathematically meaningful [11].

Graph operations provide a natural way to test the structural response of such indices. Albertson irregularity has been investigated under standard graph operations [3], and subdivision-based constructions have been studied explicitly [4]. The sigma irregularity:

$$\sigma(G) = \sum_{uv \in E(G)} \left(d_G(u) - d_G(v)\right)^2$$

Has its own extremal and structural literature [12] and has likewise been evaluated for subdivision and r-subdivision graphs [5]. Total irregularity has also been studied under subdivision-type operations [6, 10]. These papers establish that subdivision and irregularity are not new topics by themselves. In addition, the phrase partial edge-subdivision has already been used as a graph transformation in irregularity-oriented work [7]. More general questions of structural sensitivity of degree-based indices and generalized Albertson-type measures have also been considered [8, 9].

The gap addressed here is narrower. To the best of our knowledge, no exact response law has been reported for reciprocal irregularity when an arbitrary selected set of original edges is subdivided. In particular, the combination of a local endpoint-degree formula, an additive partial-subdivision law, equality conditions, invariance under insertion of multiple degree-two

vertices on the same selected edge, and the corresponding edge-selection extremals was not found. The objective of this paper is to establish that package and then compare its response with Albertson and sigma irregularity under exactly the same local transformation. The comparison is useful because it separates two measures that are monotone under subdivision from a third measure that can decrease.

2. Materials and Methods

This section fixes the notation for partial edge subdivision and records the proof and verification protocol used throughout. The key observation is that subdividing an original edge preserves the degrees of all original vertices and introduces only degree-two internal vertices, allowing each response to be localized to the replaced edge.

2.1 Graph notation and partial edge subdivision

All graphs are finite, simple and connected. For $v \in V(G)$, the degree of v is denoted by $d_G(v)$, or simply $d(v)$ when the graph is clear. Let $F \subseteq E(G)$. For every selected edge $e = uv \in F$, choose an integer $k_e \geq 1$ and replace uv by a path:

$$u = x_0 - x_1 - \cdots - x_{k_e} - x_{k_e+1} = v$$

Where the k_e internal vertices are new and have degree two. The resulting graph is denoted by $S_F^{\mathbf{k}}(G)$, where $\mathbf{k} = (k_e : e \in F)$. When $k_e = 1$ for every $e \in F$, this is the ordinary one-step partial subdivision of the selected edge set. If $F = E(G)$ and every $k_e = 1$, the usual subdivision graph $S(G)$ is obtained.

The three edge-based measures considered are:

$$\mathrm{irr}_r(G) = \sum_{uv \in E(G)} \left| \frac{1}{d(u)} - \frac{1}{d(v)} \right|$$

$$\mathrm{irr}(G) = \sum_{uv \in E(G)} |d(u) - d(v)|$$

And

$$\sigma(G) = \sum_{uv \in E(G)} \big(d(u) - d(v)\big)^2$$

For a proof device, not as a proposed new invariant, we also use:

$$I_\phi(G) = \sum_{uv \in E(G)} \left|\phi\big(d(u)\big) - \phi\big(d(v)\big)\right|$$

Where ϕ is any real-valued function on the positive integers.

2.2 Proof strategy and computational verification

Subdivision of an original edge does not change the degree of either original endpoint: one incident edge is removed and one incident edge to the first new vertex is added. Every inserted vertex has degree two. Thus all effects can be localized to the replaced edge and its new path. This observation is used repeatedly below.

The symbolic proofs were independently checked computationally. A Python verifier using NetworkX and exact *Fraction* arithmetic for reciprocal values enumerated all connected graphs in the NetworkX Graph Atlas through order seven. Every original edge was subdivided individually; repeated subdivisions with two, three and five inserted vertices were also tested. For selective subdivision, all edge subsets were exhausted for graphs with at most eight edges; for larger Atlas graphs the empty set, the full set and twenty deterministic sampled subsets were checked. The code, run manifest and frozen outputs accompany the manuscript.

3. Results and Discussion

We first derive a general local response identity and then specialize it to reciprocal irregularity. The resulting additive law yields repeated-subdivision invariance and edge-selection extremals, after which the reciprocal response is compared directly with Albertson and sigma irregularity and illustrated on a small set of graph families.

3.1 Local and reciprocal response under subdivision

Theorem 1: Let $e = uv$ be an edge of G, let $a = d_G(u)$ and $b = d_G(v)$, and let e be replaced by a path containing at least one new degree-two vertex. Then:

$$I_\phi(G') - I_\phi(G) = \Delta_\phi(a, b)$$

Where;

$$\Delta_\phi(a,b) = |\phi(a) - \phi(2)| + |\phi(b) - \phi(2)| - |\phi(a) - \phi(b)|$$

The value is independent of the number of inserted vertices. Moreover, $\Delta_\phi(a,b) \geq 0$, with equality if and only if $\phi(2)$ lies in the closed interval with endpoints $\phi(a)$ and $\phi(b)$.

Proof: The degrees of all original vertices are unchanged. The old edge contribution is $|\phi(a) - \phi(b)|$. After replacement, the first and last path edges contribute $|\phi(a) - \phi(2)|$ and $|\phi(b) - \phi(2)|$, while every internal edge joins two degree-two vertices and contributes zero. All other graph edges retain their original endpoint degrees and hence their original contributions. This proves the formula and also shows that the number of internal degree-two vertices is irrelevant. The nonnegativity is the triangle inequality on the real line. Equality in that triangle inequality occurs exactly when $\phi(2)$ lies between $\phi(a)$ and $\phi(b)$.

The theorem immediately extends from one edge to any selected original edge set because the original endpoint degrees remain fixed even when selected edges share an endpoint.

Corollary 1: For $F \subseteq E(G)$,

$$I_\phi\left(S_F^{\mathbf{k}}(G)\right) = I_\phi(G) + \sum_{uv \in F} \Delta_\phi\left(d_G(u), d_G(v)\right)$$

In particular, the total change depends on which original edges are selected, but not on the positive integers k_e.

We use this functional only as a unifying proof device; no novelty is claimed for the functional itself. The contribution concerns the resulting reciprocal and comparative response laws.

Set $\phi(x) = 1/x$. For an edge uv, let

$$a = d_G(u), \qquad b = d_G(v), \qquad s = \min\{a, b\}.$$

Theorem 2: The change in reciprocal irregularity caused by subdividing uv is:

$$\Delta_r(a,b) = \begin{cases} 1, & a = b = 1, \\ 0, & s = 1 < \max\{a,b\} \\ 1 - \dfrac{2}{s}, & s \geq 2. \end{cases}$$

Consequently, reciprocal irregularity never decreases under edge subdivision. If $G \neq K_2$, the change is positive exactly when both endpoints of the subdivided edge have degree at least three; otherwise it is zero.

Proof: Assume without loss of generality that $a \leq b$. If $a = b = 1$, then:

$$\Delta_r(1,1) = \left|1 - \frac{1}{2}\right| + \left|1 - \frac{1}{2}\right| = 1$$

If $a = 1$ and $b \geq 2$, then $1/b \leq 1/2 \leq 1$, so $1/2$ lies between the two reciprocal endpoint values and Theorem 1 gives equality, hence $\Delta_r = 0$.

Finally, if $a \geq 2$, then $1/b \leq 1/a \leq 1/2$ and:

$$\Delta_r(a,b) = \left(\frac{1}{2} - \frac{1}{a}\right) + \left(\frac{1}{2} - \frac{1}{b}\right) - \left(\frac{1}{a} - \frac{1}{b}\right) = 1 - \frac{2}{a}$$

Since $a = s$, the displayed formula follows. It vanishes when $s = 2$ and is positive when $s \geq 3$. In a connected graph an edge with both endpoint degrees one occurs only in K_2.

Define the reciprocal subdivision weight of an original edge by:

$$w_r(uv) = \Delta_r\left(d_G(u), d_G(v)\right)$$

Corollary 2: For arbitrary $F \subseteq E(G)$ and arbitrary $k_e \geq 1$,

$$\mathrm{irr}_r\left(S_F^{\mathbf{k}}(G)\right) = \mathrm{irr}_r(G) + \sum_{e \in F} w_r(e)$$

Thus inserting one degree-two vertex or many degree-two vertices on the same selected original edge has exactly the same reciprocal-irregularity effect.

For G ≠ K2, the total change for a selected set F is positive if and only if F contains at least one edge whose two original endpoints both have degree at least three.
This provides a precise refinement interpretation. Reciprocal irregularity is insensitive to subdivision of an edge touching a degree-one or degree-two vertex, except for the isolated K2 case, but it detects subdivision of edges whose two original endpoints have degree at least three.

3.2 Partial and repeated subdivision with edge-selection extremals

The additive law has an immediate extremal consequence.
Corollary 3: Fix an integer q with $0 \le q \le |E(G)|$. Among all sets $F \subseteq E(G)$ of size q, reciprocal irregularity after partial subdivision is maximized by choosing q edges with the largest values of $w_r(e)$ and minimized by choosing q edges with the smallest values of $w_r(e)$.
Proof: The base value $\mathrm{irr}_r(G)$ is independent of F. By Corollary 2, only the sum of the q selected edge weights varies. The maximum and minimum sums are obtained by the q largest and q smallest weights, respectively.
Degrees and edge weights can be computed in linear time in the graph size, and sorting all m edge weights costs O(m log m). The optimization rule is therefore an immediate structural consequence of the additive local response law.

3.3 Comparative response of reciprocal, Albertson and sigma irregularity

For Albertson irregularity, replacing uv by a path through a degree-two vertex gives:

$$\Delta_A(a,b) = |a-2| + |b-2| - |a-b|$$

For sigma irregularity the corresponding change is:

$$\Delta_\sigma(a,b) = (a-2)^2 + (b-2)^2 - (a-b)^2$$

The next theorem makes the comparison exact.
Theorem 3: For every pair of positive endpoint degrees a, b,

$$\Delta_A(a,b) = 2\min\{a,b\}\,\Delta_r(a,b)$$

And

$$\Delta_\sigma(a,b) = 2(a-2)(b-2)$$

Hence reciprocal and Albertson irregularity have identical zero/positive response classifications under edge subdivision, while sigma irregularity can decrease.
Proof: Assume $a \le b$. If $a = b = 1$, then $\Delta_A = 2$ and $\Delta_r = 1$, so the first identity holds. If $a = 1 < b$, both changes are zero. If $a \ge 2$, then:

$$\Delta_A = (a-2) + (b-2) - (b-a) = 2(a-2)$$

While Theorem 2 gives $\Delta_r = (a-2)/a$. Thus $\Delta_A = 2a\Delta_r$ in every case. For sigma, direct expansion gives:

$$\Delta_\sigma = (a-2)^2 + (b-2)^2 - (a-b)^2 = 2(a-2)(b-2)$$

Table 1 summarizes the resulting response types for a connected graph.

Table 1: Local response to subdivision of an edge with endpoint degrees a, b

Endpoint-degree type	Δ_r	Δ_A	Δ_σ	Response
$(1,1)$	1	2	2	all increase; only K_2
one endpoint has degree 2	0	0	0	all unchanged
$(1,b)$, $b \ge 3$	0	0	$-2(b-2)$	sigma decreases only
$a, b \ge 3$	$1-2/s$	$2(s-2)$	$2(a-2)(b-2)$	all increase, $s = \min\{a,b\}$

This partition gives the main conceptual distinction. Reciprocal and Albertson irregularity are monotone under graph refinement by edge subdivision, and their local changes differ only by the factor $2\min\{a,b\}$. Sigma irregularity responds differently because squaring the degree difference makes a pendant-high-degree edge less imbalanced after the original edge is split into a 1-2 edge and a 2-b edge.
The same formulas are additive over a selected set and invariant under inserting more than one degree-two vertex on any selected original edge. Consequently, sigma can increase, remain fixed, or decrease under partial subdivision depending on the degree types represented in the selected set.

3.4 Full subdivision, illustrative families and verification

Taking $F = E(G)$ gives a compact reciprocal formula for the ordinary subdivision graph.

Corollary 4: For every connected graph G,

$$\mathrm{irr}_r(S(G)) = \frac{1}{2}\sum_{v\in V(G)} |d_G(v) - 2|$$

Proof: In $S(G)$, each incidence of an original vertex v with an original edge produces one edge between v and a new degree-two vertex. Therefore:

$$\mathrm{irr}_r(S(G)) = \sum_{v\in V(G)} d_G(v)\left|\frac{1}{d_G(v)} - \frac{1}{2}\right| = \frac{1}{2}\sum_{v\in V(G)} |d_G(v) - 2|$$

This full-subdivision corollary is stated only for reciprocal irregularity as part of the present contribution. Full-subdivision results for Albertson irregularity and subdivision/r-subdivision results for sigma already occur in the literature [4, 5].

A few families illustrate the general laws without turning the paper into a catalogue.

Corollary 5: Let subdivision mean arbitrary partial subdivision of original edges, with one or more inserted degree-two vertices on every selected edge.

1. For P_n, $n \geq 3$, all three measures are invariant under any such subdivision. Their values are $\mathrm{irr}_r(P_n) = 1$, $\mathrm{irr}(P_n) = 2$, and $\sigma(P_n) = 2$.
2. For C_n, all three measures remain zero.
3. For the star $K_{1,t}$, $t \geq 2$, reciprocal and Albertson irregularity are invariant. If q leaf-center edges are selected, then:

$$\sigma\left(S_F(K_{1,t})\right) = t(t-1)^2 - 2q(t-2)$$

Hence sigma strictly decreases for $t \geq 3$ and $q > 0$.

4. If G is connected and r-regular with $r \geq 3$, subdividing any q original edges gives:

$$\mathrm{irr}_r = q\left(1 - \frac{2}{r}\right), \qquad \mathrm{irr} = 2q(r-2), \qquad \sigma = 2q(r-2)^2$$

For $r = 2$, every change is zero.

Proof: The degree types in paths are $(1,2)$ and $(2,2)$; cycles have only $(2,2)$ edges; stars have only $(1,t)$ edges; and an r-regular graph has only (r,r) edges. Substitute these degree pairs into Theorems 2 and 3 and use additivity.

The star family is particularly useful because the same local refinement leaves two irregularity measures unchanged while lowering the third. Thus the disagreement is not merely a difference in scale.

The frozen verifier covered all 996 connected Graph Atlas graphs through order seven. It executed 10,664 exhaustive single-edge identity tests, 31,992 repeated-subdivision tests and 50,391 selected-edge-set tests comprising exhaustive subsets for smaller graphs and deterministic samples for larger Atlas graphs. No counterexample was found (Table 2). Reciprocal arithmetic was exact, using rational numbers rather than floating-point comparisons.

Table 2: Verification coverage

Verification component	Cases	Result
Connected Graph Atlas graphs	996	completed
Single-edge subdivision tests	10,664	no counterexample
Repeated-subdivision tests	31,992	no counterexample
Selective edge-set tests/samples	50,391	no counterexample
Recorded counterexamples to frozen theorems	0	none

The verification is not a proof; the proofs above establish the results. Its purpose is adversarial checking of degree-one, degree-two, regular, pendant and mixed-degree boundary cases. It also exposed why a naive sigma-monotonicity analogue must be rejected: for a pendant-high-degree edge $(1,b)$ with $b \geq 3$, the exact change is negative.

The literature boundary is equally important. Reciprocal irregularity is due to Chen and Liu [2]. Subdivision as an operation is classical, and partial edge subdivision has already appeared explicitly in irregularity-oriented work [7]. Albertson and sigma subdivision results precede this paper [4, 5]. Accordingly, the contribution claimed here is not a new invariant or a new operation. It is the exact reciprocal local/selective response package and the comparative theorem that classifies three irregularity measures under the same selected-edge refinement.

4. Conclusion

The response of reciprocal irregularity to partial edge subdivision is completely determined by the original endpoint degrees. The local change is always nonnegative, vanishes on every non-K_2 selected edge incident with a degree-one or degree-two

vertex, and is positive when both endpoints have degree at least three. Because original endpoint degrees are preserved, selected-edge changes add independently and are unaffected by the number of inserted degree-two vertices. This yields an immediate fixed-budget extremal rule based on sorting local edge weights.

The comparison with two established irregularity measures reveals a sharper structural point. Albertson irregularity has exactly the same local sign behavior as reciprocal irregularity, with change equal to 2 min{d(u), d(v)} times the reciprocal change. Sigma irregularity instead changes by 2(d(u)-2)(d(v)-2) and decreases on pendant-high-degree edges. Thus a simple graph refinement can be monotone for two degree-irregularity concepts while the sigma irregularity has sign-indefinite local response and can decrease.

The present study is limited to subdivision paths whose new internal vertices all have degree two and to edge-based degree irregularity measures. Natural extensions include weighted subdivisions, simultaneous local transformations that alter original degrees, and identifying broader classes of degree functions for which exact response orderings between irregularity measures can be obtained.

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