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Fixed Point Results for Interpolative Ćirić-Reich-Rus Type Contractions in Partially Ordered b -Metric Spaces with Applications to Nonlinear Fredholm Integral Equation

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Abstract

This paper establishes novel fixed point existence theorems for self-mappings satisfying an interpolative Ćirić-Reich-Rus type contraction within the framework of partially ordered b -metric spaces. To validate the theoretical result,

an analytical example is provided. Furthermore, as an application of the proposed framework, the solvability of a class of nonlinear Fredholm integral equation is investigated.

Keywords: Fixed Point, Interpolative Ćirić-Reich-Rus Type Contraction, Partially ordered b -Metric Spaces, Fredholm Integral Equation

1. Introduction and Preliminaries

Czerwik ^[4] was the first to formulate the concept of a b -metric space, which subsequently became an important setting for fixed point ($\mathcal{FP}$) theory. Numerous studies have since established $\mathcal{FP}$ results for different types of single-valued and multivalued operator in this framework ^[2, 3, 8, 10]. The study of $\mathcal{FP}$ in partially ordered metric spaces was initiated by Ran and Reurings ^[9] in 2004 and was later developed further by Nieto and López ^[7]. Several extensions and related contributions have subsequently appeared in the literature ^[1, 5, 6].

Definition 1.1 ^[4]: Consider a set $\mathcal{B} \neq \emptyset$ and a mapping $d: \mathcal{B} \times \mathcal{B} \rightarrow [0, +\infty)$. Then, d is termed as a b -metric if there exists a real constant $s \geq 1$ such that for all $x, y, z \in \mathcal{B}$, if the following three conditions hold:

$$(B1) \quad d(x, y) = 0 \Leftrightarrow x = y;$$

$$(B2) \quad d(x, y) = d(y, x);$$

$$(B3) \quad d(x, z) \leq s(d(x, y) + d(y, z)).$$

The triplet $(\mathcal{B}, d, s)$ consequently forms a b -metric space. Furthermore, if the set $\mathcal{B}$ is equipped with a partial order $\preceq$, the quadruple $(\mathcal{B}, d, s, \preceq)$ is defined as a partially ordered b -metric space.

Definition 1.2 ^[6]: Assume $(\mathcal{B}, d, s, \preceq)$ represents a partially ordered b -metric space and let $\{x_n\}$ be a sequence within this framework, then:

1. $\{x_n\}$ converges to a point x when $\lim_{n \rightarrow +\infty} d(x_n, x) = 0$;
2. $\{x_n\}$ is a Cauchy sequence if $\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = 0$;
3. The space $(\mathcal{B}, d, s)$ is strictly complete if every Cauchy sequence admits a limit within the space itself.

Definition 1.3 ^[6]: A partially ordered b -metric space $(\mathcal{B}, d, s, \preceq)$ is said to be complete whenever the associated metric d is complete.

2. Main Result

In this section, the definition of interpolative Ćirić-Reich-Rus type contraction is first formulated within the framework of partially ordered b -metric spaces. Building upon this foundational concept, several $\mathcal{FP}$ results are then established and

proved for mappings satisfying interpolative Ćirić-Reich-Rus contraction condition.

Definition 2.1: Consider a partially ordered b - metric spaces denoted by $(\mathfrak{B}, d, \mathfrak{s}, \preceq)$ with a parameter $\mathfrak{s} > 1$. A self-mapping $\mathfrak{f}: \mathfrak{B} \rightarrow \mathfrak{B}$ is referred to as interpolative Ćirić-Reich-Rus contraction, if for all comparable elements $\mathfrak{x}, \mathfrak{y} \in \mathfrak{B} \setminus \text{Fix}(\mathfrak{f})$ (that is, whenever $\mathfrak{x} \preceq \mathfrak{y}$), the following inequality holds:

$$d(\mathfrak{f}\mathfrak{x}, \mathfrak{f}\mathfrak{y}) \leq \lambda [d(\mathfrak{x}, \mathfrak{y})]^a [d(\mathfrak{x}, \mathfrak{f}\mathfrak{x})]^b [d(\mathfrak{y}, \mathfrak{f}\mathfrak{y})]^{1-a-b} \quad (1)$$

Provided that $\lambda \in [0, 1)$ and $a, b \in (0, 1)$ where $a + b < 1$.

Theorem 2.1: Let $(\mathfrak{B}, d, \mathfrak{s}, \preceq)$ denote a complete partially ordered b - metric space with a parameter $\mathfrak{s} > 1$. Suppose that a self-mapping $\mathfrak{f}: \mathfrak{B} \rightarrow \mathfrak{B}$ is continuous, non-decreasing with respect to $\preceq$, and satisfies interpolative Ćirić-Reich-Rus contractive condition. Under these conditions, if we can find an initial element $\mathfrak{x}_0 \in \mathfrak{B}$ satisfying $\mathfrak{x}_0 \preceq \mathfrak{f}\mathfrak{x}_0$, it is guaranteed that a $\mathcal{FP}$ for $\mathfrak{f}$ exists within $\mathfrak{B}$.

Proof: Assume that $\mathfrak{x}_0 \in \mathfrak{B}$ satisfying $\mathfrak{f}\mathfrak{x}_0 = \mathfrak{x}_0$. In this case, the desired conclusion follows immediately. Otherwise, suppose that $\mathfrak{x}_0 < \mathfrak{f}\mathfrak{x}_0$, and define a sequence $\{\mathfrak{x}_n\} \subset \mathfrak{B}$ recursively by $\mathfrak{x}_{n+1} = \mathfrak{f}\mathfrak{x}_n$, ($n \geq 0$). Since, $\mathfrak{f}$ is non-decreasing, an induction yield:

$$\mathfrak{x}_0 < \mathfrak{f}\mathfrak{x}_0 = \mathfrak{x}_1 \preceq \cdots \preceq \mathfrak{f}\mathfrak{x}_n = \mathfrak{x}_{n+1} \preceq \cdots \quad (2)$$

If $\mathfrak{x}_{n_0} = \mathfrak{x}_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then by (2), $\mathfrak{x}_{n_0}$ constitutes a $\mathcal{FP}$ of $\mathfrak{f}$. Therefore, we may proceed under the assumption that $\mathfrak{x}_n \neq \mathfrak{x}_{n+1}$ for every $n \geq 1$, and hence $\mathfrak{x}_n > \mathfrak{x}_{n-1}$. Applying contraction condition (1) to these consecutive terms we have.

$$\begin{aligned} d(\mathfrak{x}_{n+1}, \mathfrak{x}_n) &= d(\mathfrak{f}\mathfrak{x}_n, \mathfrak{f}\mathfrak{x}_{n-1}) \leq \lambda [d(\mathfrak{x}_n, \mathfrak{x}_{n-1})]^a [d(\mathfrak{x}_n, \mathfrak{f}\mathfrak{x}_n)]^b [d(\mathfrak{x}_{n-1}, \mathfrak{f}\mathfrak{x}_{n-1})]^{1-a-b} \\ &= \lambda [d(\mathfrak{x}_n, \mathfrak{x}_{n-1})]^a [d(\mathfrak{x}_n, \mathfrak{x}_{n+1})]^b [d(\mathfrak{x}_{n-1}, \mathfrak{x}_n)]^{1-a-b} \\ &\leq \lambda [d(\mathfrak{x}_n, \mathfrak{x}_{n-1})]^a [d(\mathfrak{x}_n, \mathfrak{x}_{n+1})]^b [d(\mathfrak{x}_{n-1}, \mathfrak{x}_n)]^{1-a-b}, \quad (\text{for } \mathfrak{s} \geq 1) \\ &= \lambda [d(\mathfrak{x}_n, \mathfrak{x}_{n-1})]^{1-b} [d(\mathfrak{x}_n, \mathfrak{x}_{n+1})]^b \\ &\Rightarrow [d(\mathfrak{x}_n, \mathfrak{x}_{n+1})]^{1-b} \leq \lambda [d(\mathfrak{x}_{n-1}, \mathfrak{x}_n)]^{1-b} \\ &\Rightarrow d(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq \lambda d(\mathfrak{x}_{n-1}, \mathfrak{x}_n), \text{ for all } n \geq 0. \text{ Then, we have} \\ d(\mathfrak{x}_n, \mathfrak{x}_{n+1}) &\leq \lambda^{\frac{1}{1-b}} d(\mathfrak{x}_{n-1}, \mathfrak{x}_n), \text{ for all } n \geq 1 \end{aligned} \quad (3)$$

Based on the lemma from [6], the condition (3) ensures that $\{\mathfrak{x}_n\}$ forms a Cauchy sequence within the ordered b - metric space. Due to completeness of the space, $\{\mathfrak{x}_n\}$ must converge to a point $\mathfrak{f} \in \mathfrak{B}$ such that $\lim_{n \rightarrow +\infty} \mathfrak{x}_n = \mathfrak{f}$. Now, taking $\mathfrak{x} = \mathfrak{x}_n$ and $\mathfrak{y} = \mathfrak{f}$ in (1) we get:

$$d(\mathfrak{x}_{n+1}, \mathfrak{f}) = d(\mathfrak{f}\mathfrak{x}_n, \mathfrak{f}\mathfrak{f}) \leq \lambda [d(\mathfrak{x}_n, \mathfrak{f})]^a [d(\mathfrak{x}_n, \mathfrak{f}\mathfrak{x}_n)]^b [d(\mathfrak{f}, \mathfrak{f}\mathfrak{f})]^{1-a-b} \quad (4)$$

By letting $n \rightarrow +\infty$ in (4), we get $d(\mathfrak{f}, \mathfrak{f}\mathfrak{f}) = 0$. Thus, we arrive at $\mathfrak{f} = \mathfrak{f}\mathfrak{f}$. Hence, $\mathfrak{f}$ possesses a $\mathcal{FP}$.

3. Application to Nonlinear Fredholm Integral Equation

The primary objective of this section is to prove the existence of solution of Fredholm integral equations. Consider the complete partially ordered b -metric space $(\mathfrak{B}, d, \mathfrak{s}, \preceq)$, where $\mathfrak{B} = C([0, 1], \mathbb{R})$ is the set of real continuous functions on $[0, 1]$, and the metric is given by $d(\mathfrak{x}, \mathfrak{y}) = \sup_{t \in [0, 1]} |\mathfrak{x}(t) - \mathfrak{y}(t)|$ with usual order $\preceq$ i.e. $\mathfrak{x} \preceq \mathfrak{y}$ if and only if $\mathfrak{x} \leq \mathfrak{y}$. We investigate the integral equation as:

$$\mathfrak{x}(t) = \int_0^h \mathfrak{F}(t, u, \mathfrak{x}(u)) du + q(t), \quad t \in [0, 1] \quad (5)$$

Here, $q: [0, 1] \rightarrow \mathbb{R}$ and $\mathfrak{F}: [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous. We construct a mapping $\mathfrak{f}: \mathfrak{B} \rightarrow \mathfrak{B}$ defined by:

$$\tilde{f}x(t) = \int_0^h \mathfrak{F}(t, u, x(u)) du + q(t), \quad t \in [0, 1] \quad (6)$$

Clearly, a $\mathcal{FP}$ of (6) is a solution of (5).

Theorem 3.1: Let there exists $\tau > 0$ and suppose that for all $t, u, h \in [0, 1]$ and $x, y \in \mathfrak{B}$, the following relation holds true:

$$|\mathfrak{F}(t, u, x(u)) - \mathfrak{F}(t, u, y(u))| \leq e^{-\tau} |x(t) - y(t)|^a |x(t) - \tilde{f}x(t)|^b |y(t) - \tilde{f}y(t)|^{1-a-b}$$

Where $a, b, c \in (0, 1)$ such that $a + b + c < 1$. Then the integral equation (6) has a solution.

Proof: Let $x, y \in \mathfrak{B}$ and for all $t, u, h \in [0, 1]$, we have:

$$\begin{aligned} |f_x(t) - f_y(t)| &= \left| \int_0^h \mathfrak{F}(t, u, x(u)) du + q(t) - \left(\int_0^h \mathfrak{F}(t, u, y(u)) du + q(t) \right) \right| \\ &= \left| \int_0^h (\mathfrak{F}(t, u, x(u)) - \mathfrak{F}(t, u, y(u))) du \right| \\ &\leq \int_0^h |\mathfrak{F}(t, u, x(u)) - \mathfrak{F}(t, u, y(u))| du \\ &\leq e^{-\tau} |x(t) - y(t)|^a |x(t) - \tilde{f}x(t)|^b |y(t) - \tilde{f}y(t)|^{1-a-b} \end{aligned} \quad (7)$$

Letting supremum of $[0, 1]$ on both sides of (7), we get:

$$d(x, y) \leq e^{-\tau} [d(x, y)]^a [d(x, \tilde{f}x)]^b [d(y, \tilde{f}y)]^{1-a-b}$$

Hence, by Theorem 2.1, $\tilde{f}$ possesses a $\mathcal{FP}$. Thus, (5) has a solution.

4. Conclusion

This manuscript establishes foundational $\mathcal{FP}$ results for self-mappings that satisfy interpolative Ćirić-Reich-Rus type contraction condition within partially ordered b -metric spaces. The methodology not only broadens the existing mathematical framework for contractive operators but also serves as a rigorous analytical tool for applied mathematics and engineering. This dual theoretical and practical value was evidenced in our application section, where the derived theorem successfully guaranteed the existence of solutions to nonlinear Fredholm integral equation.

5. References

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