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Existence of Fixed Point for Generalized Weak ψ -Contraction Mapping in Partially Ordered b-Metric spaces with Applications to Nonlinear Volterra Integral Equation

Liza Hazarika
PhD Research Scholar, Department of Mathematics, Dibrugarh University, Dibrugarh-786004, Assam, India

Corresponding Author: **Liza Hazarika**

Abstract
This paper establishes novel fixed point existence theorems for self-mappings satisfying generalized weak ψ -contraction within the framework of partially ordered b-metric spaces. Furthermore, to validate the theoretical result, an application to a class of nonlinear Volterra integral equation is provided.

Keywords: Fixed Point, Generalized Weak ψ -Contraction, Partially Ordered b-Metric Spaces, Volterra Integral Equation

1. Introduction and Preliminaries

Czerwik [4] was the first to formulate the concept of a b-metric space, which subsequently became an important setting for fixed point ($\mathcal{FP}$) theory. Numerous studies have since established $\mathcal{FP}$ results for different types of single-valued and multivalued operator in this framework [2, 3, 8, 10]. The study of $\mathcal{FP}$ in partially ordered metric spaces was initiated by Ran and Reurings [9] in 2004 and was later developed further by Nieto and López [7]. Several extensions and related contributions have subsequently appeared in the literature [1, 5, 6].

Definition 1.1 [4]**:** Consider a set $\mathfrak{B} \neq \emptyset$ and a mapping $d: \mathfrak{B} \times \mathfrak{B} \to [0, +\infty)$. Then, d is termed as a b- metric if there exists a real constant $\mathscr{s} \geq 1$ such that for all $\mathfrak{x}, \mathfrak{y}, \mathfrak{z} \in \mathfrak{B}$, if the following three conditions hold:

(B1) $d(\mathfrak{x}, \mathfrak{y}) = 0 \Leftrightarrow \mathfrak{x} = \mathfrak{y}$;

(B2) $d(\mathfrak{x}, \mathfrak{y}) = d(\mathfrak{y}, \mathfrak{x})$;

(B3) $d(\mathfrak{x}, \mathfrak{z}) \leq \mathscr{s}(d(\mathfrak{x}, \mathfrak{z}) + d(\mathfrak{z}, \mathfrak{y}))$.

The triplet $(\mathfrak{B}, d, \mathscr{s})$ consequently forms a b-metric space. Furthermore, if the set $\mathfrak{B}$ is equipped with a partial order $\preccurlyeq$, the quadruple $(\mathfrak{B}, d, \mathscr{s}, \preccurlyeq)$ is defined as a partially ordered b-metric space.

Definition 1.2 [6]**:** Assume $(\mathfrak{B}, d, \mathscr{s}, \preccurlyeq)$ represents a partially ordered b-metric space and let $\{x_{\mathfrak{n}}\}$ be a sequence within this framework, then:

1. $\{x_{\mathfrak{n}}\}$ converges to a point $\mathfrak{x}$ when $\lim_{\mathfrak{n}\to+\infty} d(x_{\mathfrak{n}}, \mathfrak{x}) = 0$;
2. $\{x_{\mathfrak{n}}\}$ is a Cauchy sequence if $\lim_{\mathfrak{n},\mathfrak{m}\to+\infty} d(x_{\mathfrak{n}}, x_{\mathfrak{m}}) = 0$;
3. the space $(\mathfrak{B}, d, \mathscr{s})$ is strictly complete if every Cauchy sequence admits a limit within the space itself.

Definition 1.3 [6]**:** A partially ordered b-metric space $(\mathfrak{B}, d, \mathscr{s}, \preccurlyeq)$is said to be complete whenever the associated metric d is complete.

2. Main Result

In this section, the definition of generalized weak ψ-contraction is first formulated within the framework of partially ordered b- metric spaces. Building upon this foundational concept, several fixed point results are then established and proved for mappings satisfying generalized weak ψ -contraction condition.

Consider the set of function:

$$\Psi = \{\psi |\ \psi: [0, +\infty) \to [0, +\infty), \text{such that } \psi \text{ is continuous, non} - \text{decreasing, and } \psi(0) = 0\}.$$

Definition 2.1: Consider a partially ordered b- metric spaces denoted by $(\mathfrak{B}, d, s, \preccurlyeq)$ with a parameter $s > 1$. A self-mapping $\mathfrak{f}: \mathfrak{B} \to \mathfrak{B}$ is referred to as generalized weak ψ -contraction, if for all comparable elements $\mathfrak{x}, \mathfrak{y} \in \mathfrak{B} \backslash Fix(\mathfrak{f})$ (that is, whenever $\mathfrak{x} \preccurlyeq \mathfrak{y}$) and $\psi \in \Psi$, the following inequality holds:

$$s d(\mathfrak{f}\mathfrak{x}, \mathfrak{f}\mathfrak{y}) \leq \mathcal{M}(\mathfrak{x}, \mathfrak{y}) - \psi\big(\mathcal{M}(\mathfrak{x}, \mathfrak{y})\big) \tag{1}$$

Where;

$$\mathcal{M}(\mathfrak{x}, \mathfrak{y}) = \max\left\{d(\mathfrak{x}, \mathfrak{y}), \frac{d(\mathfrak{x}, \mathfrak{f}\mathfrak{x}) + d(\mathfrak{y}, \mathfrak{f}\mathfrak{y})}{1 + d(\mathfrak{x}, \mathfrak{y})}\right\}.$$

Theorem 2.1: Let $(\mathfrak{B}, d, s, \preccurlyeq)$ denote a complete partially ordered b- metric space with a parameter $s > 1$. Suppose that a self-mapping $\mathfrak{f}: \mathfrak{B} \to \mathfrak{B}$ is continuous, non-decreasing with respect to $\preccurlyeq$, and satisfies generalized weak ψ -contractive condition. Under these conditions, if we can find an initial element $\mathfrak{x}_0 \in \mathfrak{B}$ satisfying $\mathfrak{x}_0 \preccurlyeq \mathfrak{f}\mathfrak{x}_0$, it is guaranteed that a $\mathcal{FP}$ for $\mathfrak{f}$ exists within $\mathfrak{B}$.

Proof: Assume that $\mathfrak{x}_0 \in \mathfrak{B}$ satisfying $\mathfrak{f}\mathfrak{x}_0 = \mathfrak{x}_0$. In this case, the desired conclusion follows immediately. Otherwise, suppose that $\mathfrak{x}_0 \prec \mathfrak{f}\mathfrak{x}_0$, and define a sequence $\{\mathfrak{x}_\mathfrak{n}\} \subset \mathfrak{B}$ recursively by $\mathfrak{x}_{\mathfrak{n}+1} = \mathfrak{f}\mathfrak{x}_\mathfrak{n}$, ($\mathfrak{n} \geq 0$). Since, $\mathfrak{f}$ is non-decreasing, an induction yield:

$$\mathfrak{x}_0 \prec \mathfrak{f}\mathfrak{x}_0 = \mathfrak{x}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{f}\mathfrak{x}_\mathfrak{n} = \mathfrak{x}_{\mathfrak{n}+1} \preccurlyeq \cdots \tag{2}$$

If $\mathfrak{x}_{\mathfrak{n}_0} = \mathfrak{x}_{\mathfrak{n}_0+1}$ for some $\mathfrak{n}_0 \in \mathbb{N}$, then by (2), $\mathfrak{x}_{\mathfrak{n}_0}$ constitutes a $\mathcal{FP}$ of $\mathfrak{f}$. Therefore, we may proceed under the assumption that $\mathfrak{x}_\mathfrak{n} \neq \mathfrak{x}_{\mathfrak{n}+1}$ for every $\mathfrak{n} \geq 1$, and hence $\mathfrak{x}_\mathfrak{n} \succ \mathfrak{x}_{\mathfrak{n}-1}$. Applying contraction condition (1) to these consecutive terms we have,

$$d(\mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{x}_\mathfrak{n}) = d(\mathfrak{f}\mathfrak{x}_\mathfrak{n}, \mathfrak{f}\mathfrak{x}_{\mathfrak{n}-1}) < s d(\mathfrak{f}\mathfrak{x}_\mathfrak{n}, \mathfrak{f}\mathfrak{x}_{\mathfrak{n}-1})$$

$$\leq \mathcal{M}(\mathfrak{x}, \mathfrak{y}) - \psi\big(\mathcal{M}(\mathfrak{x}, \mathfrak{y})\big) \tag{3}$$

Also, from (3), we have,

$$d(\mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{x}_\mathfrak{n}) = d(\mathfrak{f}\mathfrak{x}_\mathfrak{n}, \mathfrak{f}\mathfrak{x}_{\mathfrak{n}-1}) \leq \frac{1}{s} \mathcal{M}(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}) \tag{4}$$

Where;

$$\mathcal{M}(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}) = \max\left\{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}), \frac{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{f}\mathfrak{x}_\mathfrak{n}) + d(\mathfrak{x}_{\mathfrak{n}-1}, \mathfrak{f}\mathfrak{x}_{\mathfrak{n}-1})}{1 + d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1})}\right\}$$

$$= \max\left\{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}), \frac{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1}) + d(\mathfrak{x}_{\mathfrak{n}-1}, \mathfrak{x}_\mathfrak{n})}{1 + d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1})}\right\}$$

$$= \max\{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}), d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1})\} \tag{5}$$

Suppose in (5) $\max\{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}), d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1})\} = d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1})$ for some $\mathfrak{n} \geq 1$. Under this condition, relation (4) simplifies to:

$$d(\mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{x}_\mathfrak{n}) \leq \frac{1}{s} d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1})$$

Which yield a clear contradiction since $s > 1$. Consequently, we must have $\max\{d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1}), d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1})\} = d(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}-1})$ for all $\mathfrak{n} \geq 1$. Inequality (4) then leads directly to:

$$d(\mathfrak{x}_{\mathfrak{n}+1}, \mathfrak{x}_\mathfrak{n}) \leq \frac{1}{s} d(\mathfrak{x}_{\mathfrak{n}-1}, \mathfrak{x}_\mathfrak{n})$$

Given that $\frac{1}{s} \in (0,1)$, established criteria [6] guarantee that $\{\mathfrak{x}_\mathfrak{n}\}$ forms a Cauchy sequence within the ordered b- metric space. Due to completeness of the space, $\{\mathfrak{x}_\mathfrak{n}\}$ must converge to a point $\mathfrak{k} \in \mathfrak{B}$ such that $\lim_{\mathfrak{n}\to+\infty} \mathfrak{x}_\mathfrak{n} = \mathfrak{k}$. Applying the continuity of $\mathfrak{f}$, we deduce:

$$\mathfrak{f}\mathfrak{k} = \mathfrak{f}\left(\lim_{\mathfrak{n}\to+\infty} \mathfrak{x}_\mathfrak{n}\right) = \lim_{\mathfrak{n}\to+\infty} \mathfrak{f}\mathfrak{x}_\mathfrak{n} = \lim_{\mathfrak{n}\to+\infty} \mathfrak{x}_{\mathfrak{n}+1} = \mathfrak{k}$$

Thus, we arrive at $\mathfrak{x} = \mathfrak{f}\mathfrak{x}$. Hence, $\mathfrak{f}$ possesses a $\mathcal{FP}$.

3. Application to Nonlinear Volterra Integral Equation

The primary objective of this section is to prove the existence of solution of Volterra integral equations. Consider the complete partially ordered b-metric space $(\mathfrak{B}, d, s, \preccurlyeq)$, where $\mathfrak{B} = C([0,1], \mathbb{R})$ is the set of real continuous functions on $[0,1]$, and the metric is given by $d(\mathfrak{x},\mathfrak{y}) = \sup_{t\in[0,1]} |\mathfrak{x}(t) - \mathfrak{y}(t)|$ with usual order $\preccurlyeq$ i.e. $\mathfrak{x} \preccurlyeq \mathfrak{y}$ if and only if $\mathfrak{x} \leq \mathfrak{y}$.We investigate the integral equation as:

$$\mathfrak{x}(t) = \int_0^t \mathfrak{F}(t, u, \mathfrak{x}(u))du + \mathfrak{q}(t), \quad t \in [0,1] \tag{6}$$

Here, $\mathfrak{q}: [0,1] \to \mathbb{R}$ and $\mathfrak{F}: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous. We construct a mapping $\mathfrak{f}: \mathfrak{B} \to \mathfrak{B}$ defined by:

$$\mathfrak{f}\mathfrak{x}(t) = \int_0^t \mathfrak{F}(t, u, \mathfrak{x}(u))du + \mathfrak{q}(t), \quad t \in [0,1] \tag{7}$$

Clearly, a $\mathcal{FP}$ of (7) is a solution of (6).

Theorem 3.1: Let there exists $\tau > 0$, $s > 1$ and suppose that for all $t, u \in [0,1]$ and $\mathfrak{x}, \mathfrak{y} \in \mathfrak{B}$, the following relation holds true:

$$|\mathfrak{F}(t, u, \mathfrak{x}(u)) - \mathfrak{F}(t, u, \mathfrak{y}(u))| \leq \frac{e^{-\tau}}{s} |\mathfrak{x}(t) - \mathfrak{y}(t)|$$

Where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in (0,1)$ such that $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} < 1$. Then the integral equation (6) has a solution.

Proof: Let $\mathfrak{x}, \mathfrak{y} \in \mathfrak{B}$ and for all $t, u \in [0,1]$, we have,

$$|\mathfrak{f}\mathfrak{x}(t) - \mathfrak{f}\mathfrak{y}(t)| = \left|\int_0^t \mathfrak{F}(t, u, \mathfrak{x}(u))du + \mathfrak{q}(t) - \left(\int_0^t \mathfrak{F}(t, u, \mathfrak{y}(u))du + \mathfrak{q}(t)\right)\right|$$

$$= \left|\int_0^t \left(\mathfrak{F}(t, u, \mathfrak{x}(u)) - \mathfrak{F}(t, u, \mathfrak{y}(u))\right) du\right|$$

$$\leq \int_0^t |\mathfrak{F}(t, u, \mathfrak{x}(u)) - \mathfrak{F}(t, u, \mathfrak{y}(u))| du$$

$$\leq \frac{e^{-\tau}}{s} |\mathfrak{x}(t) - \mathfrak{y}(t)| \tag{8}$$

Letting supremum of $[0,1]$ on both sides of (8), we get,

$$d(\mathfrak{f}\mathfrak{x}, \mathfrak{f}\mathfrak{y}) \leq \frac{e^{-\tau}}{s} d(\mathfrak{x}, \mathfrak{y}) \leq \frac{e^{-\tau}}{s} \mathcal{M}(\mathfrak{x}, \mathfrak{y})$$

$$= \mathcal{M}(\mathfrak{x}, \mathfrak{y}) - \left(\mathcal{M}(\mathfrak{x}, \mathfrak{y}) - e^{-\tau}\mathcal{M}(\mathfrak{x}, \mathfrak{y})\right)$$

$$s d(\mathfrak{f}\mathfrak{x}, \mathfrak{f}\mathfrak{y}) \leq \mathcal{M}(\mathfrak{x}, \mathfrak{y}) - \psi(\mathcal{M}(\mathfrak{x}, \mathfrak{y}))$$

Where $\mathcal{M}(\mathfrak{x}, \mathfrak{y}) = \max\left\{d(\mathfrak{x}, \mathfrak{y}), \frac{d(\mathfrak{x},\mathfrak{f}\mathfrak{x}) + d(\mathfrak{y},\mathfrak{f}\mathfrak{y})}{1 + d(\mathfrak{x},\mathfrak{y})}\right\}$ and $\psi \in \Psi$ such that $\psi(t) = t - e^{-\tau}t$. Hence, by Theorem 2.1, $\mathfrak{f}$ possesses a $\mathcal{FP}$. Thus, (6) has a solution.

4. Conclusion

This manuscript establishes foundational $\mathcal{FP}$ results for self-mappings that satisfy generalized weak ψ -contractive condition within partially ordered b-metric spaces. The methodology not only broadens the existing mathematical framework for contractive operators but also serves as a rigorous analytical tool for applied mathematics and engineering. This dual theoretical and practical value was evidenced in our application section, where the derived theorem successfully guaranteed the existence of solutions to nonlinear Volterra integral equation.

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