Int. j. adv. multidisc. res. stud. 2026; 6(4):838-841

Received: 03-06-2026
Accepted: 13-07-2026

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

Properties of D–Ordered Solutions of the Pell Equation

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DOI: https://doi.org/10.62225/2583049X.2026.6.4.6659

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Abstract
Minimal solutions of the Pell Equation according to the List provided by A.H. Beiler are analyzed and Equivalence classes are derived. Particular cases of numerical relationships are studied.

Keywords: Pell Equation, Equivalence Classes, Numerical Constrains

1. Introduction
An apparently simple Diophantine equation [2], whose formal expression is:

$$x^2 - Dy^2 = 1 \qquad (1)$$

Where x and y are integers and D is a natural number, which is not a square, was intensively studied by mathematicians of India starting in the 7th century, [5, 15, 11], who developed methods to obtain solutions of the equation and showed that it had infinite many solutions. Ten centuries later, in the 17th century, this equation arrived to Europe and attracted the interest of mathematicians like Fermat, Lagrange, Brouncker, and Euler [7, 15]. It is known [15] that Fermat challenged his colleagues to solve the Pell Equation " $x^2 - 61y^2 = 1$ ", which, as we will indicate below, is a difficult problem. Somehow erroneously, the equation was said to have been introduced by the mathematician, John Pell [6], and the wrong name "Pell Equation" [4, 13] has remained until our days.
This equation is important in Number Theory [8] and the search for methods to solve it, is still a challenging subject. It is known that the Indian mathematician Brahmagupta, for instance, developed methods to solve this equation and some of its generalizations [15]. In present days, the method of continued fractions of $\sqrt{D}$, [1, 16] possibly first suggested by Brouckner and Euler. was formally further developed by Lagrange, and has become a classical one to solve the equation [14, 15]. This powerful method to solve the Pell Equation is however not the only one. We are interested in "indirect" methods, which allow to obtain new solutions based on other known solutions [9, 10, 14].
The study of the Pell Equation has developed in different directions. A rather unusual one is the result of work done by A.H. Beiler [3, 14], who published a list of minimal solutions for the Pell Equation, ordered by increasing integer values of the parameter D between 2 and 102, excluding squares. In what follows we show that there are equivalence classes in the list, which allow obtaining sets of solutions from each class. But also it is known that there are "unique" elements in the list, with unusual large minimal entries.
In former works [9, 10] we used the notation (x, D, y) to represent a solution of the Pell Equation, possibly, because of the similarity with the structure of (1). In what follows we will use the notation (D, x, y), which follows the structure of the entries of the List of A.H. Beiler [3, 14].

2. Equivalence Classes
The following Lemmas consider $2 \leq D \leq 102$, excluding squares, as in [3, 14].
Lemma 1: Let $k \in \mathbb{N}\backslash\{1\}$. If $D = k^2 - 1$, then $(D, x, y) = (k^2 - 1, k, 1)$ is a minimal solution and if $D = k^2 + 1$ then:
$(k^2 + 1, 2k^2 + 1, 2k)$ is also a minimal solution.
Proof: If (D, x, y) is a solution of the Pell Equation, then $D = (x^2 - 1)/y^2$ [9].
If $D = k^2 - 1$ then clearly x = k and y = 1. Therefore $(k^2 - 1, k, 1)$ is a minimal solution (as shown in [3]).
If $D = k^2 + 1$ then $(x^2 - 1)/y^2 = k^2 + 1$. This is a (new) Diophantinc equation. Let y = 2k. Then:

$(x^2 - 1) = (4k^2)(k^2 + 1) = 4k^4 + 4k^2$

$x^2 = 4k^4 + 4k^2 + 1 = (2k^2 + 1)^2$

$x = 2k^2 + 1$

Therefore $(D, x, y) = (k^2 + 1, 2k^2 + 1, 2k)$ is a minimal solution (as shown in [3]). □

Examples:

k^2	$D = k^2-1$	$x = k$	$y = 1$	$D = k^2+1$	$x = 2k^2+1$	$y = 2k$
4	3	2	1	5	9	4
9	8	3	1	10	19	6
16	15	4	1	17	33	8
25	24	5	1	26	51	10
36	35	6	1	37	73	12
49	48	7	1	50	99	14
64	63	8	1	65	129	16
81	80	9	1	82	163	18
100	99	10	1	101	201	20

Let α denote the class based on $D = k^2 - 1$ and β denote the class based on $D = k^2 + 1$.

Lemma 2: Let $k \in \mathbb{N}\backslash\{1\}$. If $D = k^2 + 2$, then $(D, x, y) = (k^2 + 2, k^2 + 1, k)$ is a minimal solution.
Proof: Since if (D, x, y) is a solution of the Pell Equation, then $D = (x^2 - 1)/y^2$ [9],
For the Lemma $D = ((k^2 + 1)^2 - 1)/k^2 = (k^4 + 1 + 2k^2 - 1) / k^2 = (k^4 + 2k^2) / k^2 = k^2 + 2$ □
The solutions obtained are minimal since for all $2 \leq k \leq 10$ they appear in the list of Beiler [3, 14].
Let χ denote this class.

Example:

$D = k^2 + 2$	$x = k^2 + 1$	$y = k$
6	5	2
11	10	3
18	17	4
27	26	5
38	37	6
51	50	7
66	65	8
83	82	9
102	101	10

Lemma 3: Let $k \in \mathbb{N}\backslash\{1\}$. If (D, x, y) is a minimal solution according to [3], then $(k^2 - 2, k^2 - 1, k)$ is also a minimal solution.
Proof: See Lemma 2 of [9]. □

Examples:

$D = k^2 - 2$	$x = k^2 - 1$	$y = k$
2	3	2
7	8	3
14	15	4
23	24	5
34	35	6
47	48	7
62	63	8
79	80	9
98	99	10

Let δ denote the class characterized by $D = k^2 - 2$.

Lemma 4: Let $k \in \mathbb{N}\backslash\{1\}$. If (D, x, y) is a minimal solution according to [3], then $(4k^2 - 4, 2k^2 - 1, k)$ is also a minimal solution.
Proof: Exhaustive search in the list [3] and summarized below. Numbers in parenthesis correspond to valid solutions. Which are however beyond Beiler's list. □

$D = 4k^2 - 4$	$x = 2k^2 - 1$	$y = k$
12	7	2
32	17	3
60	31	4
96	49	5
(140)	(71)	(6)
(192)	(97)	(7)
(252)	(127)	(8)
(320)	(161)	(9)
(396)	(199)	(10)

Let ε denote this class.

Lemma 5: Let $k \in \mathbb{N}\backslash\{1\}$. If (D, x, y) is a minimal solution according to [3], then:
$(4(k^2 + 1), 2k^2 + 1, k)$ is also a minimal solution.
Proof: Exhaustive search in the list [3] and summarized below. □

$D = 4k^2 + 4$	$x = 2k^2 + 1$	$y = k$
20	9	2
40	19	3
68	33	4
(104)	(51)	(5)

Let ϕ denote this class. The numbers in parenthesis correspond to a correct solution, however beyond Beiler's list. Notice that the solution (D, x, y) = (8, 3, 1) of class α could also be ordered in this class.

Remark: Classes may be partially related. For example, Class ϕ including the solution (8, 3, 1) may be associated to the even D entries of class χ.

Class f			Partial Class c with D even		
D	x	y	$D_c = D_f - 2$	$x_c = 2x_f - 1$	$y_c = 2y_f$
8	3	1	6	5	2
20	9	2	18	17	4
40	19	3	38	37	6
68	33	4	66	65	8
(104)	(51)	(5)	102	101	10

Moreover, some solutins may belong to different classes, depending of the relative interpretation of the relationship between D, x and y. As mentioned above, the solution (8, 3, 1) may be included in both the classes α and ϕ. The solution (12, 7, 2) of class ε under the interpretation $(D = 4k^2 - 4, x = 2k^2 - 1, y = k)$ generates a new class under the interpretation $(D = k(k^3 - 2), x = k^3 - 1, y = k)$:

$D = k(k^3 - 2)$	$x = k^3 - 1$	$y = k$
12	7	2
75	26	3
(248)	(63)	(4)

Let this new class be called γ.

It is also possible to build classes whose elements are *pairs of solutions*, which otherwise belong to different "individual" classes.

Lemma 6: Let $k \in \mathbb{N}\backslash\{1\}$. If (D, x, y) is a minimal solution according to [3], then:

both $(D = k^2 + 2, x = k^2 + 1, y = k)$

and $(D = k^2 + 1, x = 2k^2 + 1, y = 2k)$

Are minimal solutions and constitute a single element of a new class. Notice that the value of x of the first solution equals the value of D of the second one!

Proof: Exhaustive search in the list [3] and summarized below. □

Example:

$(D = k^2 + 2, x = \underline{k^2 + 1}, y = k)$	$(D = \underline{k^2 + 1}, x = 2k^2 + 1, y = 2k)$
6, 5, 2	5, 9, 4
11, 10, 3	10, 19, 6
18, 17, 4	17, 33, 8
27, 26, 5	26, 51, 10
38, 37, 6	37, 73, 12
51, 50, 7	50, 99, 14
66, 65, 8	65, 129, 16
83, 82, 9	82, 163, 18
102, 101, 10	101, 201, 20

It is simple to check that this class brings together the classes β and χ.

Lemma 7: A rather unusual class comprises solutions where y is a prime, $D = y^2 - 2$, and $x = y^2 - 1$.
Proof: From (2), $D = (x^2 - 1)/y^2$. Introducing the defining conditions, $x^2 - 1 = y^4 - 2y^2 = y^2(y^2 - 2)$, from where:

$(x^2 - 1)/y^2 = y^2 - 2 = D$ □

(Notice that in the given examples below, except for D = 287. D is also a prime)

D	x	y
2	3	2
7	8	3
23	24	5
47	48	7
(119)	(120)	(11)
(167)	(168)	(13)
(287)	(288)	(17)
(359)	(360)	(19)

This class is actually a sub-class of class δ.

Fig 1 Shows row-wise all D prameters of the Beiler List. Shaded grey are all square values, which D should not take. Places with greek letters identify the classes related to the former Lemmas. The places with a star denote two special cases with unusual relationships, as will be illustrated below. The remaining places correspond to solutions with increasing numerical entries, which however could not be ordered in classes.

D	0	1	2	3	4	5	6	7	8	9
0			δ	α		β	χ	δ	α	
1	β	χ	ε	☆	δ	α		β	χ	
2	ϕ	η	η	δ	α		β	χ		
3			ε		δ	α		β	χ	
4	ϕ							δ	α	
5	β	χ					η			
6	ε	☆	δ	α		β	χ		ϕ	
7		η				γ				δ
8	α		β	χ	η				η	
9							ε		δ	α
10		β	χ		(ϕ)					

Fig 1: Classes of minimal solutions of the Pell Equation for values of D from 2 to 102. Not labelled are solutions with incresing numerical values of x and y

3. Solutions with related numerical entries

It should be recalled that from (1), $x^2 - Dy^2 = 1$ follows that:

$$D = \frac{x^2 - 1}{y^2} \qquad (2)$$

From (2) quite obviously follows that changes in y induce oposite changes in D. Particulartly if y is reduced by ½, D will be scaled by 4 and no changes will affect x. Considering this in the context of the list of Beiler [3, 14], a new class, called h, is obtained, which, as in the class folowing Lemma 6, each entry consists of a pair of solutions of the Pell Equation.
Proof: Exhaustive search in the list [3] and summarized below. □

Example:

Original			New		
D	x	y	4D	x	y/2
2	3	2	8	3	1
5	9	4	20	9	2
6	5	2	24	5	1
10	19	6	40	19	3
13	649	180	52	649	90
14	15	4	56	15	2
17	33	8	68	33	4
18	17	4	72	17	2
21	55	12	84	55	6
22	197	42	88	197	21
26	51	10	(104)	51	5

Recall that Beiler's List consists of minimal solutions of the Pell Equation when D ≤ 102. Beyond the above examples with y/2, except for the pair of solutions (10, 19, 6) and (90, 19, 2), where y is reduced to y/3, and then D increases by 9, no other cases comprising larger fractions of y were found.
Notice that the fact that in this class the values of x are preserved for pairs of solutions explains why the pairs of solutions with D = 13 and D = 52 as well as pairs of solutions with D = 22 and D = 88, preserve the unexpected large x-entries 649 and 197, respectively.
Of particular interest in this study are the solutions (13, 649, 180) and (61, 1.766.319.049, 226.153.980).
To alleviate the presentation, in what follows we may use the following simplified notation:

$(13, x_{13}, y_{13})$ and $(61, x_{61}, y_{61})$.

First of all, both 13 and 61 are prime numbers. (x_{13} is a semiprime since $x_{13} = 649 = 11{\cdot}59$).
$(13, x_{13}, y_{13})$ is the solution of the Pell Equation with smallest values x_{13}, y_{13} larger than 100.
$(61, x_{61}, y_{61})$ is the (minimal) solution with the largest values of x_{61} and y_{61} in the list of Beiler [3, 14].
Notice that although 13 and 61 are obviously relative primes, $x_{61} = x_{13} \cdot 2.721.601$ and $y_{61} = y_{13} \cdot 1.256.411$.
These are quite unexpected relationships. (Finally, notice that 1.256.411 = **13** × 127 × 761).
Another interesting solution is (73, 2.281.249, 267.000) since 2.281.249 is a prime! whereas 267.000 has four prime factors: 2, 3, 5 and 89.
It should be mentioned that the Indian mathematician Bhaskara II developed the first general method to solve the Pell Equation in the year 1150 [12]. Bhaskara II used his method to solve, among others, the difficult Equation for D = 61 mentioned above.

4. Closing Remarks

The study of solutions of the Pell Equation, ordered by increasing non-square positive values of D –(as presented in the Beiler's List)– has shown that there are a good number of equivalence classes relating solutions. On the other hand, the requirement that D must be a non-square positive integer, as shown in (2), leads to several unusual integer numerical values of x and y, which may be obtained e.g., with the method of continued fractions of $\sqrt{D}$, [1, 16].
Unexpected relationships between the solutions for D = 13 and D = 61 have been found, where the solution with D = 61 requires the largest values for x and y in the List [3].

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