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Algebraic Aspects of Solutions of the Pell Equation

Claudio Moraga
Technical University of Dortmund, Otto-Hahn-Str. 12, 44127, Dortmund, Germany

Corresponding Author: **Claudio Moraga**

Abstract
This work shows that powers of any solution of the Pell Equation lead to new solutions of this Equation. The new solutions preserve the variable y and associates the power to the variable x. As otherwise required, the parameter D satisfies the quotient $(x^2 - 1)/y^2$. Furthermore, it is shown that the product of (different) solutions of the Pell Equation do not provide new solutions. Similarly, the average of any two different solutions of the Pell Equation does not constitute a new solution.

Keywords: Pell Equation, Diophantine equations, Germany

1. Introduction

Diophantine equations [1, 12] constitute an important subject of number theory [2, 3, 9]. Among Diophantine equations, a most relevant is the one called (in Europe) Pell Equation [4, 12, 13] defined as:

$$x^2 - Dy^2 = 1 \tag{1}$$

(In the European notation), Where x and y are integers, and D is a non-square natural number. Since, as may be seen in (1) x and y are squared, it is enough to consider x any as natural numbers for related calculations. Several European mathematicians are known to have contributed to develop methods to solve the equation and determine its main properties. Among the best known are, Fermat, Lagrange [8] and Euler, (See [14]). It is said, [5], that Euler by mistake considered the mathematician John Pell [6] to be the one that had presented the Equation (1) and this is how this equation became the name "Pell". See [14] for details of the Pell equation and its main extensions.
Equation (1) had been however studied much earlier, (*around 1000 years before Pell's time* [15]) under the notation:

$$x^2 - N y^2 = 1 \tag{2}$$

By mathematicians from India Among the best known are, Brahmagupta, Bhaskara II and Narayama Pandit. A comprehensive description of the works of these mathematicians may be found in [15].

2. Powers of solutions of the Pell Equation

Lemma 1: If (x, D, y) denotes a solution of the Pell Equation, then $x^2 - Dy^2 = 1$. It is simple to see that also $(x^2 - Dy^2)^2 = 1$. Moreover, it has the structure of a Pell solution. It preserves y and squares x.
Proof: Since $x^2 - Dy^2 = 1$, $(x^2 - Dy^2)^2 = 1$

$$(x^2 - Dy^2)^2 = x^4 + D^2y^4 - 2Dx^2y^2 = 1$$
$$x^4 - D(2x^2y^2 - Dy^4) = 1$$
$$(x^2)^2 - D(2x^2 - Dy^2)y^2 = 1$$

Let $D' = D(2x^2 - Dy^2)$. Then $(x^2)^2 - D'y^2 = 1$

From where (x^2, D', y) is a (new) solution of the Pell Equation, which clearly preserves y and squares x. □

Example 1: Consider the Pell solution (24, 23, 5)

$D' = D(2x^2 - Dy^2) = 23(2\times576 - 23\times25) = 23(1.152 - 575) = 13.271$,

$(x^2)^2 - D'y^2 = 331.776 - 13.271\times25 = 331.776 - 331.775 = 1$

i.e., $(x^2, 13.271, y) = (576, 13.271, 5)$ is a new solution of the Pell Equation.

Example 2: Consider the Pell solution (161, 320, 9)

$D' = D(2x^2 - Dy^2) = 320 \times (2 \times 161^2 - 320 \times 81) = 16.589.440 - 8.294.400 = 8.295.040$

$(x^2)^2 - D'y^2 = 25.921^2 - 8.295.040\times81 = 671.898.241 - 671.898.240 = 1$

i.e., $(x^2, 8.295.040, y) = (576, 8.295.040, 9)$ is a new solution of the Pell Equation.

Lemma 2: If (x, D, y) is a solution of the Pell Equation, then $x^2 - Dy^2 = 1$. It is simple to see that also: $(x^2 - Dy^2)^3 = 1$. Moreover, it has the structure of a Pell solution.

Proof: Since $x^2 - Dy^2 = 1$ then $(x^2 - Dy^2)^3 = 1$.

$(x^2 - Dy^2)^3 = x^6 - 3Dx^4y^2 + 3D^2x^2y^4 - D^3y^6 = x^6 - (3Dx^4 - 3D^2x^2y^2 + D^3y^4) \times y^2 = 1$

$D' := 3Dx^4 - 3D^2x^2y^2 + D^3y^4$.

Then $(x^3)^2 - D'y^2 = 1$,

from where (x^3, D', y) is a new solution of the Pell Equation.

Example 3: Consider the Pell solution (50, 51, 7)

$D' = 3Dx^4 - 3D^2x^2y^2 + D^3y^4 = 318.877.551$

$(x^2)^3 = 15.625\times10^6$

$x^6 - D'y^2 = 15.625.000.000 - 318.877.551\times49 = 15.625.000.000 - 15.624.999.999 = 1$

Therefore, $(x^3, D', y) = (125.000, 318.887.551, 7)$ is a new solution of the Pell Equation.

Theorem 1: Let (x, D, y) be a solution of the Pell Equation. Then $x^2 - Dy^2 = 1$. It is simple to see that for any integer *n* also $(x^2 - Dy^2)^n = 1$. Let it be assumed that from $(x^2 - Dy^2)^n = 1$ follows $((x^n)^2 - D'y^2) = 1$ where,

$D' = ((x^n)^2 - 1)/y^2$. Then for *any* $m > n$, $(x^2 - Dy^2)^m = 1$.

Proof: by induction [7]

$x^2 - Dy^2 = 1$

$(x^2 - Dy^2)^n = 1$

$(x^2 - Dy^2)^n = ((x^n)^2 - D'y^2) = 1$

$(x^2 - Dy^2)^{n+1} = ((x^n)^2 - D'y^2)(x^2 - Dy^2) = 1$

$= (x^{n+1})^2 - ((x^n)^2Dy^2) - (x^2D'y^2) + DD'y^4 = 1$

$= (x^{n+1})^2 - (((x^n)^2D - x^2D' + DD'y^2)\, y^2) = 1$

$= (x^{n+1})^2 - D''y^2$, where $D'' = ((x^n)^2D - x^2D' + DD'y^2)$

i.e., For any $m > n$ if $x^2 - Dy^2 = 1$ then $(x^2 - Dy^2)^m = 1 \Rightarrow ((x^m)^2 - D''y^2) = 1$ and $((x^m), D'', y)$ is a new solution.

Example 4: Given a general solution (x, D, y) of the Pell Equation, then $x^2 - Dy^2 = 1$. Consider the case:

$(x^2 - Dy^2)^4 = 1$:

$(x^2 - Dy^2)^4 = x^8 - 4Dx^6y^2 + 6D^2x^4y^4 - 4D^3x^2y^6 + D^4y^8 = 1$

$= x^8 - (4Dx^6 + 6D^2x^4y^2 - 4D^3x^2y^4 + D^4y^6)y^2 = 1$

$= (x^4)^2 - D'y^2 = 1$ with $D' = -4Dx^6 + 6D^2x^4y^2 - 4D^3x^2y^4 + D^4y^6$

If the solution (7, 3, 4) is chosen, then $x^4 = 2.401$, $x^8 = 5.764.801$ and $y^2 = 16$.

$D' = (4\times27\times49\times256 - 6\times9\times2.401\times\hat{}16 + 4\times3\times117.649 - 81\times4.096)$

$= (1.354.752 - 2.074.464 + 1.411.788 - 331.776) = 360.300$

$x^8 - D'y^2 = 5.764.801 - 360.300 \times 16 = 5.764.801 - 5.764.800 = 1$

3. Analysis of possible generalizations

G1: Let (x_1, D_1, y_1) and (x_2, D_2, y_2) be different solutions of the Pell Equation.

Then $(x_1)^2 - D_1(y_1)^2 = 1$ and $(x_2)^2 - D_2(y_2)^2 = 1$.

As a simple generalization of Lemma 1, instead of considering the squares of the former equations, let us take their product. Clearly,

$\langle (x_1)^2 - D_1(y_1)^2 \rangle \times \langle (x_2)^2 - D_2(y_2)^2 \rangle = 1$

$(x_1)^2(x_2)^2 - (x_1)^2D_2(y_2)^2 - (x_2)^2D_1(y_1)^2 + D_1D_2(y_1)^2(y_2)^2 = 1$

$(x_1)^2(x_2)^2 - (x_1)^2D_2(y_2)^2 - (x_2)^2D_1(y_1)^2 - D_1D_2(y_1)^2(y_2)^2 + 2\, D_1D_2(y_1)^2(y_2)^2 = 1$

To obtain a solution of the Pell Equation from this equation, two conditions are necessary:

i) $2\, D_1D_2(y_1)^2(y_2)^2 - (x_1)^2D_2(y_2)^2 - (x_2)^2D_1(y_1)^2 = 0$

ii) $(x_1)^2(x_2)^2 - D_1D_2(y_1)^2(y_2)^2 = 1$. Notice that this condition may be directly obtained with a Hadamard-type product of the two considered solutions.

Test. Let (3, 2, 2) and (7, 3, 4) be solutions from the Pell Equation [10].

Then $((x_1)^2 - 2(y_1)^2)((x_2)^2 - 3(y_2)^2) = 1$

$(x_1)^2(x_2)^2 - 3(x_1)^2(y_2)^2 - 2(x_2)^2(y_1)^2 + 6(y_1)^2(y_2)^2 = 1$

$441 - 432 - 392 + 384 = 1$

To be a solution of the Pell Equation it is necessary that there exist D' such that:

$(x_1)^2(x_2)^2 - D'(y_1)^2(y_2)^2 = 1$

i.e., $441 - D'64 = 1$, but since D' must equal $(x^2 - 1)/y^2$ [10], then $D' = 440/64$ leading to $D' = 6{,}875\ldots$

But D' should be a positive non-square integer [13]. Therefore, the intended generalization does not hold.

However, it is important to point out that the Indian mathematician Brahmagupta discovered a new equivalent extended expression to the above product of the two basic solutions, which is today known as "Brahmagupta's identity". This allowed him to find solutions of a "generalized equation" $x^2 - Dy^2 = k$ and with these, to find infinitely many solutions of the original (2) Equation. A detailed explanation is offered in [15].

G2: Let again (x_1, D_1, y_1) and (x_2, D_2, y_2) be different solutions of the Pell Equation.
Then $(x_1)^2 – D_1(y_1)^2 = 1$ and $(x_2)^2 – D_2(y_2)^2 = 1$.

It becomes apparent that the average of both solutions also equals 1. Is it however a new solution of the Pell Equation?

$$(1/2)((x_1)^2 – D_1(y_1)^2 + (x_2)^2 – D_2(y_2)^2) = 1.$$

$$(1/2)((x_1)^2 + (x_2)^2 – D_1(y_1)^2 – D_2(y_2)^2) = 1.$$

To be a solution of the Pell Equation it would be needed that $(x_1)^2 + (x_2)^2 = (x_3)^2$ and, moreover, $(x_3)^2$ should be even, in which case this requirement implies that (x_1, x_2, x_3) must be a Pythagorean triple. However, this is not possible since $(x_3)^2$ even requires that x_1 and x_2 have the same parity, but for a Pythagorean triple, x_1 and x_2 must have opposite parity [11]. Therefore, the intended "additive" generalization does also not hold.

4. Closing Remark

Results presented in this paper may be considered as an extension of those of [10] and are intended to motivate further study of properties of the solutions of the Pell Equation.

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