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Coefficient Estimates and Fekete-Szegö Problem for the Pseudo-Starlike and Pseudo-Convex Univalent Function Classes Related with $q-$ Derivative

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Abstract

In this study, we define a new subclass of starlike and convex univalent functions defined in the open unit disk of the complex plane. Here, we give some upper bounds for the coefficients and solve Fekete-Szegö problem for this newly defined class.

Keywords: Pseudo-starlike function, pseudo-convex function, $q-$ derivative

1. Introduction

In this section, we give some basic information which we will use in our study.

Let $H(U)$ be the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ of the complex plane $\mathbb{C}$. By A, we will denote class of the functions $f\in H(U)$ given by the following series expansions:

$$f(z)=z+a_2z^2+a_3z^3+a_4z^4+\cdots+a_nz^n+\cdots=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C} \tag{1.1}$$

It is clear that the function $f\in H(U)$ belonging to the class A satisfies the conditions $f(0)=0$ and $f'(0)=1$.

The subclass of univalent functions of A is denoted by S in the literature. This class was first introduced by Köebe [1] and has become the central object of study in geometric function theory. Later, many mathematicians became interested in coefficient estimates for this class. Within a short period, in 1916 Bieberbach [2] published a paper in which the famous coefficient hypothesis was proposed. This hypothesis states that if $f\in S$ and has the series form (1.1), then $|a_n|\le n$ for each $n\ge 2$. This coefficient hypothesis is known as the Bieberbach conjecture in the literature. In 1985, it was de-Branges [3], who settled this long-lasting conjecture. There were a lot of papers devoted to this conjecture in the literature (see [4-14]).

It is well known that the starlike and convex function classes in the open unit disk U are denoted by S^* and C and defined analytically as follows, respectively:

$$S^*=\left\{f\in S:\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in U\right\},\ C=\left\{f\in S:\ \operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)>0,\ z\in U\right\}$$

Let's $f,g\in H(U)$, then it is said that f is subordinate to g and denoted by $f\prec g$, if there exists a Schwartz function ω, such that $f(z)=g(\omega(z))$.

In the past few years, numerous subclasses of the class S have been introduced and studied, as special choices of the classes S^* and C (see for example [4, 9, 10, 15-25]).

2. Materials and Methods

Throughout this section and further on, we always make use the classical definitions of quantum concepts as follows.

The $q-$numbers and $q-$factorials are defined, respectively by:

$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1} = \sum_{k=0}^{n-1} q^k, q \in (0,1), n \in \mathbb{N};$

It is well known that in the standard approach to the $q-$ calculus $q-$ exponential function defined as follows:

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{n=0}^{\infty} \frac{1}{[1-(1-q)q^n z]}, \ 0<|q|<1, \ |z|<\frac{1}{|1-q|}$$

It is clear that $\lim_{q\to 1}[n]_q = n$ and $\lim_{q\to 1}[n]_q! = n!$. Also, $\lim_{q\to 1} e_q^z = e^z$.

Now we define new subclass of univalent functions defined in the open unit disk U.

Definition 2.1: For $q\in(0,1)$, $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau \in \mathbb{C}-\{0\}$ the function $f\in S$ is said to be in the class $\chi(\tau,\lambda,\beta;e_q^z)$, if the following condition is satisfied:

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z(f'(z))^\lambda}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[z(f'(z))'\right]^\lambda}{f'(z)}-1\right]\right\}\prec e_q^z, \ z\in U$$

In the cases $\beta=0$, $\beta=1$ and $\lambda=1$ from the Definition 2.1, we have the following classes of univalent functions.

Definition 2.2: For $q\in(0,1)$, $\lambda>\frac{1}{2}$ and $\tau \in \mathbb{C}-\{0\}$ the function $f\in S$ is said to be in the class $S^*(\tau,\lambda;e_q^z)$, if the following condition is satisfied:

$$1+\frac{1}{\tau}\left[\frac{z(f'(z))^\lambda}{f(z)}-1\right]\prec e_q^z, \ z\in U$$

Definition 2.3: For $q\in(0,1)$, $\lambda>\frac{1}{2}$ and $\tau \in \mathbb{C}-\{0\}$ the function $f\in S$ is said to be in the class $C(\tau,\lambda;e_q^z)$, if the following condition is satisfied:

$$1+\frac{1}{\tau}\left[\frac{\left[z(f'(z))'\right]^\lambda}{f'(z)}-1\right]\prec e_q^z, \ z\in U$$

Definition 2.4: For $q\in(0,1)$, $\beta\in[0,1]$ and $\tau \in \mathbb{C}-\{0\}$ the function $f\in S$ is said to be in the class $\chi(\tau,\beta;e_q^z)$, if the following condition is satisfied:

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z(f'(z))}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{z(f'(z))'}{f'(z)}-1\right]\right\}\prec e_q^z, \ z\in U$$

In addition, in the case $q\to 1^-$ from the Definition 2.1, we have the following class of univalent functions.

Definition 2.5: For $\beta\in[0,1]$, $\lambda>\frac{1}{2}$ and $\tau \in \mathbb{C}-\{0\}$ the function $f\in S$ is said to be in the class $\chi(\tau,\lambda,\beta;e^z)$, if the following condition is satisfied:

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{z(f'(z))^\lambda}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{\left[z(f'(z))'\right]^\lambda}{f'(z)}-1\right]\right\}\prec e^z, \ z\in U$$

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\operatorname{Re}(p(z))>0$, $z\in U$ It is clear that the functions that satisfy these conditions have the following series expansion:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n, \; z \in U \tag{2.1}$$

The class P defined above is known as the class Caratheodory functions in the literature [26].
Now, let us give some necessary lemmas for the proof of our main results.

Lemma 2.1 ([27]). Let the function p belong to the class P. Then,
$|p_n| \le 2$ for each $n \in \mathbb{N}$, $|p_n - \nu p_k p_{n-k}| \le 2$ for $n, k \in \mathbb{N}$, $n > k$ and $\nu \in [0,1]$.
The equalities hold for the function:

$$p(z) = \frac{1+z}{1-z}$$

Lemma 2.2 ([27]) Let the analytic function p be of the form (2.1), then:

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)y$$

For some $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.
In this paper, we give some coefficient estimates and solve Fekete-Szegö problem for the class $\chi(\tau, \lambda, \beta; e_q^z)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results obtained in the literature.

3. Results and Discussion

In this section, we give some coefficient estimates for the functions belonging to the class $\chi(\tau, \lambda, \beta; e_q^z)$ and solve Fekete-Szegö problem for this class.
First of all, we give the following theorem for coefficient estimates.
Theorem 3.1: Let the function f given by series expansions (1.1) belong to the class $\chi(\tau, \lambda, \beta; e_q^z)$. Then, we have the following upper bound estimates:

$$|a_2| \le \frac{|\tau|}{(2\lambda - 1)(1+\beta)}$$

and

$$|a_3| \le \frac{|\tau|}{(3\lambda-1)(1+2\beta)} \begin{cases} 1 & \text{if } a(q,\tau,\lambda,\beta) \le 1, \\ a(q,\tau,\lambda,\beta) & \text{if } a(q,\tau,\lambda,\beta) \ge 1, \end{cases} \tag{3.1}$$

Where,

$$a(q,\tau,\lambda,\beta) = \frac{\left|(2\lambda^2 - 4\lambda + 1)(1+3\beta)(1+q)\tau - (2\lambda-1)^2(1+\beta)^2\right|}{(2\lambda-1)^2(1+\beta)^2}$$

Proof: Let $f \in \chi(\tau, \lambda, \beta; e_q^z)$, then exists Schwartz function $\omega : U \to U$, such that

$$(1-\beta)\left\{1 + \frac{1}{\tau}\left[\frac{z(f'(z))^{\lambda}}{f(z)} - 1\right]\right\} + \beta\left\{1 + \frac{1}{\tau}\left[\frac{\left[(zf'(z))'\right]^{\lambda}}{f'(z)} - 1\right]\right\} = e_q^{\omega(z)}, \; z \in U \tag{3.2}$$

Let's the function $p \in P$ defined as follows:

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n, \; z \in U \tag{3.3}$$

It follows from that:

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+\frac{1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)z^3+\cdots,\ z\in U \tag{3.4}$$

From the (3.2) and (3.4), it can be written that:

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[a_2(2\lambda-1)z+\left((3\lambda-1)a_3+\left(2\lambda^2-4\lambda+1\right)a_2^2\right)z^2+\cdots\right]\right\}$$
$$+\beta\left\{1+\frac{1}{\tau}\left[2a_2(2\lambda-1)z+\left(3(3\lambda-1)a_3+4\left(2\lambda^2-4\lambda+1\right)a_2^2\right)z^2+\cdots\right]\right\}$$
$$=1+\frac{p_1}{2}z+\frac{1}{4}\left(2p_2-\frac{q}{1+q}p_1^2\right)z^2+\cdots,\ z\in U \tag{3.5}$$

By equating the coefficients on the left and right sides of the equality (3.5), we get:

$$(2\lambda-1)(1+\beta)a_2=\frac{\tau}{2}p_1;\ \text{that is}\ a_2=\frac{\tau}{2(2\lambda-1)(1+\beta)}p_1 \tag{3.6}$$

$$(3\lambda-1)(1+2\beta)a_3+\left(2\lambda^2-4\lambda+1\right)(1+3\beta)a_2^2=\frac{\tau}{4}\left(2p_2-\frac{q}{1+q}p_1^2\right) \tag{3.7}$$

Using Lemma 2.1 to the equality (3.6), we have the first result of theorem.
Now let's find an upper bound estimate for the coefficient a_3. From the equality (3.7) using the expression for a_2 (see (3.6)), we can write the following equality for the coefficient a_3:

$$a_3=\frac{\tau}{4(3\lambda-1)(1+2\beta)}\left[2p_2-\frac{\left(2\lambda^2-4\lambda+1\right)(1+3\beta)(1+q)\tau+q(2\lambda-1)^2(1+\beta)^2}{(2\lambda-1)^2(1+\beta)^2(1+q)}p_1^2\right] \tag{3.8}$$

Applying Lemma 2.2 to the equality (3.8), we can write:

$$a_3=\frac{\tau}{4(3\lambda-1)(1+2\beta)}\left[\left(4-p_1^2\right)x-\frac{\left(2\lambda^2-4\lambda+1\right)(1+3\beta)(1+q)\tau-(2\lambda-1)^2(1+\beta)^2}{(2\lambda-1)^2(1+\beta)^2(1+q)}p_1^2\right]$$

For some $x\in\mathbb{C}$ with $|x|\le 1$. Then, we have:

$$|a_3|\le\frac{|\tau|}{4(3\lambda-1)(1+2\beta)}\left[a(q,\tau,\lambda,\beta)t^2+\left(4-p_1^2\right)\xi\right],$$

Where, $\xi=|x|$, $t=|p_1|$ and,

$$a(q,\tau,\lambda,\beta)=\frac{\left|\left(2\lambda^2-4\lambda+1\right)(1+3\beta)(1+q)\tau-(2\lambda-1)^2(1+\beta)^2\right|}{(1+q)(2\lambda-1)^2(1+\beta)^2}$$

Later, by maximizing the function $\varphi:\mathbb{R}^2\to\mathbb{R}$ defined as follows, with respect to ξ:

$$\varphi(\xi,t)=a(q,\tau,\lambda,\beta)t^2+\left(4-t^2\right)\xi,\ \xi\in[0,1],\ t\in[0,2],$$

We have,

$$\varphi(\xi,t)\le\left[a(q,\tau,\lambda,\beta)-1\right]t^2+4,\ t\in[0,2]$$

From here, it is easily to see that $\varphi(\xi,t)\le 4$ if $a(q,\tau,\lambda,\beta)\le 1$ and $\varphi(\xi,t)\le 4a(\tau,\lambda,\beta)$ if $a(q,\tau,\lambda,\beta)\ge 1$.
With this, the proof of second inequality of (3.1) is provided.
Thus, the proof of the theorem is completed.
In the case $\beta=0$, $\beta=1$, $\lambda=1$ and $q\to 1^-$ from the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.1: If $f \in S^*\left(\tau, \lambda; e_q^z\right)$, then:

$$|a_2| \le \frac{|\tau|}{2\lambda - 1} \text{ and } |a_3| \le \frac{|\tau|}{3\lambda - 1}\begin{cases} 1 & if\ a(q,\tau,\lambda) \le 1, \\ a(q,\tau,\lambda) & if\ a(q,\tau,\lambda) \ge 1, \end{cases}$$

Where,

$$a(q,\tau,\lambda) = \frac{\left|\left(2\lambda^2 - 4\lambda + 1\right)(1+q)\tau - (2\lambda-1)^2\right|}{(2\lambda-1)^2}$$

Corollary 3.2: If $f \in C\left(\tau, \lambda; e_q^z\right)$, then:

$$|a_2| \le \frac{|\tau|}{2(2\lambda - 1)} \text{ and } |a_3| \le \frac{|\tau|}{3(3\lambda - 1)}\begin{cases} 1 & if\ b(q,\tau,\lambda) \le 1, \\ b(q,\tau,\lambda) & if\ b(q,\tau,\lambda) \ge 1, \end{cases}$$

Where,

$$b(q,\tau,\lambda) = \frac{\left|\left(2\lambda^2 - 4\lambda + 1\right)(1+q)\tau - (2\lambda-1)^2\right|}{(2\lambda-1)^2}$$

Corollary 3.3: If $f \in \chi\left(\tau, \beta; e_q^z\right)$, then:

$$|a_2| \le \frac{|\tau|}{1+\beta} \text{ and } |a_3| \le \frac{|\tau|}{2(1+2\beta)}\begin{cases} 1 & if\ c(q,\tau,\beta) \le 1, \\ c(q,\tau,\beta) & if\ c(q,\tau,\beta) \ge 1, \end{cases}$$

Where,

$$c(q,\tau,\beta) = \frac{\left|(1+3\beta)(1+q)\tau + (1+\beta)^2\right|}{(1+\beta)^2}$$

Corollary 3.4: If $f \in \chi\left(\tau, \lambda, \beta; e^z\right)$, then:

$$|a_2| \le \frac{|\tau|}{(2\lambda-1)(1+\beta)} \text{ and } |a_3| \le \frac{|\tau|}{(3\lambda-1)(1+2\beta)}\begin{cases} 1 & if\ d(\tau,\lambda,\beta) \le 1, \\ d(\tau,\lambda,\beta) & if\ d(\tau,\lambda,\beta) \ge 1, \end{cases}$$

Where,

$$d(\tau,\lambda,\beta) = \frac{\left|2\left(2\lambda^2 - 4\lambda + 1\right)(1+3\beta)\tau - (2\lambda-1)^2(1+\beta)^2\right|}{(2\lambda-1)^2(1+\beta)^2}$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $\chi\left(\tau, \lambda, \beta; e_q^z\right)$.

Theorem 3.2: Let $f \in \chi\left(\tau, \lambda, \beta; e_q^z\right)$ and $\mu \in \mathbb{C}$, then:

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{(3\lambda-1)(1+2\beta)}\begin{cases} 1 & if\ l(q,\tau,\lambda,\beta,\mu) \le 1, \\ l(q,\tau,\lambda,\beta,\mu) & if\ l(q,\tau,\lambda,\beta,\mu) \ge 1, \end{cases} \quad (3.9)$$

Where,

$$l(q,\tau,\lambda,\beta,\mu) = \frac{\left|(2\lambda-1)^2(1+\beta)^2 - (1+q)\left[\left(2\lambda^2 - 4\lambda + 1\right)(1+3\beta) + (3\lambda-1)(1+2\beta)\mu\right]\tau\right|}{(1+q)(2\lambda-1)^2(1+\beta)^2}$$

Proof. Let $f \in \chi\left(\tau, \lambda, \beta; e_q^z\right)$. Then, from the equality (3.7) the expression $a_3 - \mu a_2^2$ can written as follows:

$$a_3-\mu a_2^2=\left[\frac{(2\lambda^2-4\lambda+1)(1+3\beta)}{(1-3\lambda)(1+2\beta)}-\mu\right]a_2^2+\frac{\tau}{4(3\lambda-1)(1+2\beta)}\left(2p_2-\frac{q}{1+q}p_1^2\right)$$

Applying the first equality of Lemma 2.2 to this equality and considering equality (3.6), we get:

$$a_3-\mu a_2^2=\frac{\tau}{4(3\lambda-1)(1+2\beta)}\times\left\{(4-p_1^2)x+\frac{(2\lambda-1)^2(1+\beta)^2-(1+q)\left[(2\lambda^2-4\lambda+1)(1+3\beta)+(3\lambda-1)(1+2\beta)\mu\right]\tau}{(1+q)(2\lambda-1)^2(1+\beta)^2}p_1^2\right\} \quad (3.10)$$

For some $x\in\mathbb{C}$ with $|x|\le 1$.
Then, using the triangle inequality to the equality (3.10), we have:

$$|a_3-\mu a_2^2|\le\frac{|\tau|}{4(3\lambda-1)(1+2\beta)}\times\left\{(4-t^2)\xi+\frac{\left|(2\lambda-1)^2(1+\beta)^2-(1+q)\left[(2\lambda^2-4\lambda+1)(1+3\beta)+(3\lambda-1)(1+2\beta)\mu\right]\tau\right|}{(1+q)(2\lambda-1)^2(1+\beta)^2}t^2\right\} \quad (3.11)$$

Where, $\xi=|x|$ and $t=|p_1|$.
Maximizing the expression on the right side of the inequality (3.11) with respect to the variable ξ, we obtain the following inequality:

$$|a_3-\mu a_2^2|\le\frac{|\tau|}{4(3\lambda-1)(1+2\beta)}\left\{\left[l(q,\tau,\lambda,\beta,\mu)-1\right]t^2+4\right\},\ t\in[0,2]$$

Where,

$$l(q,\tau,\lambda,\beta,\mu)=\frac{\left|(2\lambda-1)^2(1+\beta)^2-(1+q)\left[(2\lambda^2-4\lambda+1)(1+3\beta)+(3\lambda-1)(1+2\beta)\mu\right]\tau\right|}{(1+q)(2\lambda-1)^2(1+\beta)^2}$$

Also, maximizing the function $\psi:[0,2]\to\mathbb{R}$ defined as follows:

$$\psi(t)=\left[l(\tau,\lambda,\beta,\mu)-1\right]t^2+4,\ t\in[0,2]$$

We can easily see that $\psi(t)\le 4$ if $l(q,\tau,\lambda,\beta,\mu)\le 1$ and $\psi(t)\le 4l(q,\tau,\lambda,\beta,\mu)$ if $l(q,\tau,\lambda,\beta,\mu)\ge 1$.
Thus, the proof of theorem is completed.
In the case $\beta=0$, $\beta=1$, $\lambda=1$ and $q\to 1^-$ from the Theorem 3.2, we obtain the following results.

Corollary 3.5: If $f\in S^*(\tau,\lambda;e_q^z)$ and $\mu\in\mathbb{C}$, then:

$$|a_3-\mu a_2^2|\le\frac{|\tau|}{3\lambda-1}\begin{cases}1 & if \quad l_0(q,\tau,\lambda,\mu)\le 1,\\ l_0(q,\tau,\lambda,\mu) & if \quad l_0(q,\tau,\lambda,\mu)\ge 1,\end{cases}$$

Where,

$$l_0(q,\tau,\lambda,\mu)=\frac{\left|(2\lambda-1)^2-(1+q)\left[(2\lambda^2-4\lambda+1)+(3\lambda-1)\mu\right]\tau\right|}{(1+q)(2\lambda-1)^2}$$

Corollary 3.6: If $f\in C(\tau,\lambda;e_q^z)$ and $\mu\in\mathbb{C}$, then:

$$|a_3-\mu a_2^2|\le\frac{|\tau|}{3(3\lambda-1)}\begin{cases}1 & if \quad l_1(q,\tau,\lambda,\mu)\le 1,\\ l_1(q,\tau,\lambda,\mu) & if \quad l_1(q,\tau,\lambda,\mu)\ge 1,\end{cases}$$

Where,

$$l_1(q,\tau,\lambda,\mu)=\frac{\left|4(2\lambda-1)^2-(1+q)\left[4\left(2\lambda^2-4\lambda+1\right)+3(3\lambda-1)\mu\right]\tau\right|}{4(1+q)(2\lambda-1)^2}$$

Corollary 3.7: If $f\in\chi\left(\tau,\beta;e_q^z\right)$ and $\mu\in\mathbb{C}$, then:

$$\left|a_3-\mu a_2^2\right|\le\frac{|\tau|}{(1+2\beta)}2\begin{cases}1 & \text{if} \quad l_2(q,\tau,\beta,\mu)\le 1,\\ l_2(q,\tau,\beta,\mu) & \text{if} \quad l_2(q,\tau,\beta,\mu)\ge 1,\end{cases}$$

Where,

$$l_2(q,\tau,\beta,\mu)=\frac{\left|(1+\beta)^2+(1+q)\left[(1+3\beta)-2(1+2\beta)\mu\right]\tau\right|}{(1+q)(1+\beta)^2}$$

Corollary 3.8: If $f\in\chi\left(\lambda,\beta;e^z\right)$ and $\mu\in\mathbb{C}$, then:

$$\left|a_3-\mu a_2^2\right|\le\frac{|\tau|}{(3\lambda-1)(1+2\beta)}\begin{cases}1 & \text{if} \quad l_3(\tau,\lambda,\beta,\mu)\le 1,\\ l_3(\tau,\lambda,\beta,\mu) & \text{if} \quad l_3(\tau,\lambda,\beta,\mu)\ge 1,\end{cases}$$

Where,

$$l_3(\tau,\lambda,\beta,\mu)=\frac{\left|(2\lambda-1)^2(1+\beta)^2-2\left[\left(2\lambda^2-4\lambda+1\right)(1+3\beta)+(3\lambda-1)(1+2\beta)\mu\right]\tau\right|}{2(2\lambda-1)^2(1+\beta)^2}$$

Taking $\mu=0$ and $\mu=1$ in the Theorem 3.2, we obtain the following results, respectively.

Corollary 3.9: If $f\in\chi\left(\tau,\lambda,\beta;e_q^z\right)$, then:

$$|a_3|\le\frac{|\tau|}{(3\lambda-1)(1+2\beta)}\begin{cases}1 & \text{if} \quad l_4(q,\tau,\lambda,\beta,\mu)\le 1,\\ l_4(q,\tau,\lambda,\beta,\mu) & \text{if} \quad l_4(q,\tau,\lambda,\beta,\mu)\ge 1,\end{cases}$$

Where,

$$l_4(q,\tau,\lambda,\beta,\mu)=\frac{\left|(2\lambda-1)^2(1+\beta)^2-(1+q)\left[\left(2\lambda^2-4\lambda+1\right)(1+3\beta)\right]\tau\right|}{(1+q)(2\lambda-1)^2(1+\beta)^2}$$

Corollary 3.10: If $f\in\chi\left(\tau,\lambda,\beta;e_q^z\right)$, then:

$$\left|a_3-a_2^2\right|\le\frac{|\tau|}{(3\lambda-1)(1+2\beta)}\begin{cases}1 & \text{if} \quad l_5(q,\tau,\lambda,\beta)\le 1,\\ l_5(q,\tau,\lambda,\beta) & \text{if} \quad l_5(q,\tau,\lambda,\beta)\ge 1,\end{cases}$$

Where,

$$l_5(q,\tau,\lambda,\beta)=\frac{\left|(2\lambda-1)^2(1+\beta)^2-(1+q)\left[\left(2\lambda^2-4\lambda+1\right)(1+3\beta)+(3\lambda-1)(1+2\beta)\right]\tau\right|}{(1+q)(2\lambda-1)^2(1+\beta)^2}$$

Remark 3.1: As can be seen that Corollary 3.9 confirms the second result of Theorem 3.1.

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