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On the Structure of Sequences of Solutions of the Pell Equation

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Abstract
In this paper the Pell Equation and a class of their solutions are studied. It is shown that sequences of solutions may be obtained when fixing one of the variables, that the first solution is simple to obtain and that the sequences have a growing structure. Explicit growing rules allow extending preliminary short sequences without limit. Thus, sequences of solutions are countably infinite.

Keywords: Pell Equation, Sequences of Solutions, Growing Rules

1. Introduction
Number theory [8] is a relevant chapter of Discrete Mathematics. The study of Diophantine Equations [2, 4, 6, 11] is one important subject of Number Theory.
Among the Diophantine Equations, the Pell Equation [3, 11] and several of its extensions have been intensively studied, possibly starting, in Europe, with the important contributions of the great mathematicians Fermat [5] and Lagrange [7]. It may be mentioned that the name "Pell Equation" is due to Leonhard Euler who erroneously attributed this equation to John Pell [5].
The (basic) Pell Equation has the following form:

$$x^2 - Dy^2 = 1, \qquad (1)$$

Where x, D, y $\in \mathbb{Z}$ and D is positive and not a square [12].

Usually in the literature on the Pell Equation (x, y) are defined as integers, If for some given D a pair of integers (x, y) provides a solution, then it becomes apparent that the pairs (x, -y), (-x, y) and (-x,-y) will also solve (1) since they appear squared. For simplicity, in what follows we restrict x and y to be natural numbers. Due to the structure of (1), the "other" solutions follow straightforwardly.
It has been shown [11, 1] that a method based on the continued fraction of the square root of D always provides a solution of the Pell Equation. Moreover, a Table of minimal solutions [12] for values of D between 2 and 101 is included in [12]. References to other tables of solutions are also presented in [12]. Additional Tables of solutions with minimal y and with minimal x may be found in [13] under Pell's equation.
In the present paper we study the Pell Equation from a different point of view. We develop small sequences (of length 10) of solutions for increasing values of y. A way of providing a minimal solution for each sequence is provided. Each sequence is ordered by increasing values of x and the corresponding values of D are calculated. The structure of the sequences is analyzed, and it is shown that they have special generation rules. The corresponding rules are determined. This allows us to explicitly extend the sequences, up to infinity. The sets of solutions are therefore, countably infinite [9]. That the Pell Equation has infinitely many solutions had already been stated by J.L. Lagrange [7], who is said to be the first *European* to solve the equation. Mathematicians from India had earlier intensively studied the Pell equation [12].

2. Formalisms
Definition 1: Infinite does not have a numerical value. It stands for "without limit".
Definition 2: [10] A set is countably infinite if it can be ordered 1 to 1 with the natural numbers.
Definition 3: For a given y, a solution of the Pell Equation with the smallest value of x will be called *minimal*.

Lemma 1: When x = y, there are no solutions for the Pell Equation.

Proof: If $x = y$ then $x^2 - Dy^2 = 1$ becomes $x^2(1 - D) = 1$. Since D must be a non-square natural number, then $D > 1$, from Where $1 - D$ is negative. A negative $1 - D$ leads to an imaginary x, but to be a solution, x must be a natural number. Therefore, to solve the Pell Equation x and y must be different.
If **y = 1**, From $x^2 - Dy^2 = 1$ follows $D = x^2 - 1$, with $x > 1$ since $x \neq y$. The solutions have the structure $(x, x^2-1, 1)$. Since x must be different from y, $x \in \mathbb{N}\backslash\{1\}$ and this set of solutions is countably infinite. Recall that "infinite" is not a number. It stands for "without limit".

Lemma 2: Let (x, D, y) be the notation to represent a solution (or a solution candidate) of the Pell Equation. Then for any $y \in \mathbb{N}\backslash\{1\}$, $(y^2 - 1, y^2 - 2, y)$ is a solution.
Proof: With $y > 1$, $x^2 - Dy^2$ becomes $(y^2 - 1)^2 - (y^2 - 2)\cdot y^2 = (y^4 - 2y^2 + 1) - (y^4 - 2y^2) = 1$ as required in (1).
Remark: Solutions obtained with Lemma 2 are not necessarily minimal unless y is odd. For example, if $y = 6$ then with Lemma 2, (35, 34, 6) is a solution, but according to [2] (in [12]), (19, 10, 6), is claimed to be a solution with minimal values of x and D. However, we proved (with Lemma 3), that (17, 8, 6) is still a better solution.

Lemma 3: Let (x, D, y) be a solution of the Pell Equation.
Then for any y even, $y > 2$ and x odd, $\langle (x - 1)/2, (D - 2)/4, y \rangle$ is also a solution.
Proof: With Lemma 2, $(x - 1)/2 = (y^2 - 2)/2$ and $(D - 2)/4 = (y^2 - 4)/4$. Then equation (1) becomes:

$$\langle (x-1)/2 \rangle^2 - \langle (D - 2)/4 \rangle \cdot y^2 = (y^2 - 2)^2 /4 - ((y^2 - 4)/4)\cdot y^2 = \langle (y^4 - 4y^2 + 4) - (y^4 - 4y^2) \rangle /4 = 1$$

Lemma 4: Let (x, D, y) be a solution of the Pell Equation with $y > 2$, but $(x - 2, D', y)$ is not a solution. Then
$(x + 2, D'', y)$ is a new solution of the Pell Equation.
Proof: From (x, D, y) follows $D = (x^2 - 1)/y^2$. If $(x+2, D'', y)$ is a solution then $D'' = ((x+2)^2 - 1)/y^2 =$

$$= (x^2 + 4 + 4x - 1)/y^2 = (x^2 - 1)/y^2 + 4(x + 1)/y^2 = D + 4(x + 1)/y^2.$$

With (1), $(x + 2)^2 - D''y^2 = x^2 + 4 + 4x - (D + 4(x + 1)/y^2)\cdot y^2 = x^2 - Dy^2 + 4 + 4x - 4 - 4x = 1$ □
With respect to the restriction that $(x - 2, D', y)$ is not a solution, let by contradiction be assumed that it is a solution. Then, $D' = ((x - 2)^2 - 1)/y^2 = (x^2 - 1 + 4 - 4x)/y^2 = D - 4(x - 1)/y^2$. D is known to be an integer and, according to the assumption D' would also be an integer. Therefore, $4(x - 1)/y^2$ should also be an integer.
Under the given assumption, let us now evaluate a candidate solution $((x + 2), D'', y)$:

$$D'' = ((x + 2)^2 - 1)/y^2 = (x^2 + 4x + 4 - 1)/y^2 = D + 4(x + 1)/y^2 = D + 4((x - 1) + 2)/y^2 = D + 4(x - 1)/y^2 + 8/y^2.$$

Since $(x - 2, D', y)$ was assumed to be a solution, D' should be an integer and from its analysis $4(x - 1)/y^2$ should also be an integer. Then $D + 4(x - 1)/y^2$ would also be an integer. However, $8/y^2$ is not an integer since $y > 2$. This means that, D'' would not be an integer and then, $(x + 2, D'', y^2)$ would not be a solution. This contradicts the Lemma. Therefore, the assumption was false.
Example: Consider first a simple case, with a small y: (26, 75, 3), where $26^2 - 75\cdot 3^2 = 676 - 675 = 1$, meaning that (26, 75, 3) is a solution of the Pell Equation. Now we check "x – 2": (24, D, 3). If this were another correct solution then D should equal $(x^2 - 1)/y^2 = 575/9$, but this is not an integer, as required for D. Therefore, (24, D, 3) is not a solution. Then we check the $x \rightarrow x+2$ transformation and try (28, D', 3) and obtain $D' = (28^2 - 1)/9 = 87$. Consider now a solution with a large y: (499.999, 249.999, 1000), obtained with Lemma 3, whereas a direct calculation with "x – 2" shows that if x = 499.997 and y = 1000, then $(x^2 - 1)/y^2 = 249.997(,)000.008$ which is not an integer as required for D. Then, x = 499.997 and y = 1.000 do not provide a solution. On the other hand (500.001, 250.001, 1000) satisfies (1) since $x^2 = (500.001)^2 = 250.001.000.001$, $Dy^2 = 250.001.000.000$ and $x^2 - Dy^2 = 1$.

Lemma 5: Let (x, D, y) be a solution of the Pell Equation with y even and $y > 2$, but $(x - 2, D', y)$ is not a solution.
Then $(x + y^2/2, D^\#, y)$ is a new solution of the Pell Equation.
Proof: From (x, D, y) follows $D = (x^2 - 1)/y^2$. If $(x+ y^2/2, D^\#, y)$ is a solution then:

$$D^\# = ((x + y^2/2)^2 - 1)/y^2 = (x^2 + xy^2 + y^4/4 - 1)/y^2 = (x^2 - 1)/ y^2 + x + y^2/4 = D + x + y^2/4$$

With (1), $(x + y^2/2)^2 - (D + x + y^2/4)\cdot y^2 = x^2 + xy^2 + y^4/4 - Dy^2 - xy^2 - y^4/4 = x^2 - Dy^2 = 1$. □
Example: Consider the solution (31, 15, 8). We check now a candidate solution with "x – 2", i. e., (29, D'', 8). To be a solution, D'' should equal $(x^2 - 1)/y^2$, but $29^2 - 1 = 840$, which is not divisible by 64. Therefore (29. D'', 8) is not a solution. Now we check a new candidate with $(x + y^2/2) = 31 + 32 = 63$, i.e., (63, D', 8). In this case we have $D' = (63^2 - 1)/64 = 3.968/64 = 62$. Therefore, (63, 62, 8) is indeed a correct new solution.
With respect to the restriction that $((x - 2), D, y)$ should not be a solution, see Lemma 4.

Lemma 6: Let (x, D, y) be a solution of the Pell Equation with y even and $y > 2$, but $(x - 2, D', y)$ is not a solution.
Then $(x + y^2, D'', y)$ is a new solution of the Pell Equation.
Proof: From (x, D, y) follows $D = (x^2 - 1)/y^2$. If $(x+ y^2, D'', y)$ is a solution then:

$D'' = ((x+ y^2)^2 - 1)/y^2 = (x^2 + 2xy^2 + y^4 -1)/y^2= (x^2 – 1)/ y^2 + 2x +y^2 = D + 2x + y^2$.

With (1), $(x + y^2)^2 – (D + 2x + y^2)\cdot y^2 = x^2 + 2xy^2 + y^4 – Dy^2 – 2xy^2 – y^4 = x^2 – Dy^2 = 1$. □
Example: Consider again the solution (499.999, 249.999, 1.000) for which we showed in Lemma 4 that if x = 499.997 no solution is possible. The transformation $x_{new} \to x + y^2$ leads to x_{new} = 1.499.999, $(x_{new})^2 = 2.249.997\cdot 10^6$ and $(x_{new}^2 – 1)/y^2 = D_{new}$ = 2.249.997. Then $(x_{new})^2 – D_{new}y^2 = (x_{new})^2 – \langle(x_{new}^2 – 1)/y^2 \rangle\cdot y^2 = 1$.
With respect to the restriction that (x – 2, D', y) is not a solution, see Lemma 4.

2.1 Analysis of Cases when y is even

Case y = 2. From $x^2 – Dy^2 = 1$ follows $x^2 – 4D = 1$. It becomes apparent that x > 1 and for all D in ℕ, 4D + 1 is odd and then x must also be odd, from where the earliest solution is obtained with y = 2, x = 3 (since x cannot be equal to y), and D = 2, leading to $9 - 2\cdot 2^2 = 9 - 2\cdot 4 = 1$. Notice that the solution (3, 2, 2) is minimal and would also be obtained with Lemma 2, although y is even, but not larger than 2. Therefore, Lemma 3 could not be applied.

Lemma 7: With y = 2, if (x, D, 2) is a solution, the next solution is (x', D', 2), where x' = x + 2 and
D' = D + x + 1.
Proof: From $x^2 – Dy^2 = 1$ follows $D = (x^2 – 1)/y^2 = (x^2 – 1)/ 4$.
Then D' must satisfy $((x')^2 – 1)/4 = ((x + 2)^2 – 1)/4 = (x^2 -1 + 4x + 4)/4 = (x^2 -1)/4 + (4x + 4)/4 = D + x + 1$. □
Example: From (3, 2, 2) follows (5, 6, 2) and from (5, 6, 2) follows (7, 12, 2).
Check: $(x')^2– D'\cdot y^2 = 7^2 – 12\cdot 4 = 49 – 48 = 1$
Let 𝕆 denote the set of odd natural numbers. This set is countably infinite.
From Lemma 7, if y = 2, the solutions can be ordered without limit according to x ∊ 𝕆\{1}. Therefore, the set of solutions when y = 2 is countably infinite.

Lemma 8: Let (x, D, 2) be a solution. If D = 4d, then (x, d, y') is also a solution, where y' = 2y = 4.
Proof: From $x^2 – Dy^2 = 1$ follows $x^2 – (4d)y^2 = x^2 –(d4)y^2 = x^2 – d\cdot(2y)^2 = 1$ □
Example: As shown above, (7, 12, 2) is a solution.
Notice that (7, 12, 2) = (7, (3·4), 2). Then (7, 3, 4) is also a solution, since $7^2 – 3\cdot 16 = 1$.

Case y = 4. From Lemma 8 it may be deduced that solutions with y = 4 share a subset of the solutions with y = 2, which are also obtained "locally" with Lemmas 4, 5 and 6. The set of solutions grows (without limit). Hence the set of solutions with y = 4 is also countably infinite.
See columns y = 2 and y = 4 in Table 1. Solutions when y = 2 are ordered according to odd values (larger than 2). of x. Entries 3 and 4 when y = 2 appear as entries 1 and 2 when y = 4. Similarly, entries 7 and 8 when y = 2, appear as entries 3 and 4 when y = 4. Notice that entries 3 and 4 when y = 4 are "inherited" from entries 7 and 8 when y = 2. But they are generated by the transformation $x \to x+2$, $x \to x+y^2/2$ or $x \to x+y^2$ of Lemmas 4, 5 and 6.
A similar situation may be observed in pairs of sequences when y = 3 and y = 6 as well as when y = 4 and y = 8.

2.2 Analysis of Cases when y is odd

Case y = 3 From $x^2 – Dy^2 = 1$ follows $x^2 = 9D + 1$. With Lemma 2 the solution (8, 7, 3) is obtained and there is not a solution with x < 8. With Lemma 4, a new solution (10, 11, 3) is obtained. Further, with Lemma 5 (17, 32, 3) and again with Lemma 4, (19, 40, 3) are obtained. The same strategy leads further to (26, 75, 3) and (28, 87, 3) and supports the growth of the sequence without limit.

Lemma 9: As in Lemma 4, let (x, D, 3) be a solution, but (x-2, D', 3) not a solution. Then (x+2, D", 3) is a new solution.
Proof: From (x, D, 3) follows $D = (x^2 – 1)/y^2$. If (x+2, D", 3) is a solution then $D'' = ((x+2)^2 - 1)/9 =$
$= (x^2 + 4 + 4x - 1)/9 = (x^2 – 1)/9 + 4(x + 1)/9 = D + 4(x + 1)/9$. Then $(x + 2)^2 – 9D'' = (x + 2)^2 – 9(D + 4(x + 1)/9) = (x + 2)^2 – 9D - 4(x+1) = (x + 2)^2 – (x^2 – 1) – 4x – 4 = = x^2 – (x^2 – 1) = 1$.

Table 1: Initial sequences of solutions of the Pell Equation when y ∊ {2, 3, 4, 5, 6, 7, 8}

i	y = 2		y = 3		y = 4		y = 5		y = 6		y = 7		y = 8	
	x	D	x	D	x	D	x	D	x	D	x	D	x	D
1	3	2	8	7	7	3	24	23	17	8	48	47	31	15
2	5	6	10	11	9	5	26	27	19	10	50	51	33	17
3	7	12	17	32	15	14	49	96	35	34	97	192	63	62
4	9	20	19	40	17	18	51	104	37	38	99	200	65	66
5	11	30	26	75	23	33	74	219	53	78	146	435	95	141
6	13	42	28	87	25	39	76	231	55	84	148	447	97	147
7	15	56	35	136	31	60	99	392	71	140	195	776	127	252
8	17	72	37	152	33	68	101	408	73	148	197	792	129	260
9	19	90	44	215	39	95	124	615	89	220	244	1215	159	395
10	21	110	46	235	41	105	126	635	91	230	246	1235	161	405

With respect to the restriction that (x-2, D', 3) is not a solution, we proceed as in Lemma 4.

By contradiction let it be assumed that (x-2, D', 3) were indeed a solution. Then $(x - 2)^2 - D'y^2 = 1$ leads to $x^2 - 4x + 4 - 9D' = 1$ from where D' = D – 4(x– 1)/9. Notice that since both D and according to the assumption D' are integers, then 4(x– 1)/9 must also be an integer.
We now evaluate the candidate ((x+2), D'', 3). $D'' = ((x+2)^2 - 1)/9 = (x^2 + 4x + 4 - 1)/9 = D + 4(x + 1)/9 =$
$= D + 4(x - 1 + 2)/9 = D + 4(x - 1)/9 + 8/9$. Recall that D and 4(x– 1)/9 are integers, but 8/9 is obviously not. Therefore D'' would not be an integer and then ((x+2), D'', 3) would also not be a solution. This contradicts the Lemma. Hence the assumption was false.
Example: Consider the solution (17, 32, 3). Then x – 2 = 15, and $x^2 - 1 = 224$, which is not divisible by 9. Therefore, no integer D exists. (15, D, 3) is not a solution. Then from the solution (17, 32, 3) a new solution may be obtained with (x + 2) = 19 and D' = D + 4(x + 1)/9 = 32 + 4·18/9 = 40. Check: From (19, 40, 3) follows $19^2 - 40 \cdot 9 = 361 - 360 = 1$.
If we now start with the solution (19, 40, 9), x - 2 = 17 and (17, 32, 3) is known to be a solution, then no new solution will be obtained with x + 2. A simple calculation shows that (21, D'', 9) is not a solution since $21^2 - 1 = 440$, which is not divisible by 9.

Lemma 10: Let (x, D, 3) be a solution but, as in Lemma 6, (x-2, D, 3) is not a solution. Then $(x+y^2) = (x + 9)$ and ((x+9), D'', 3) is a new solution.
Proof: From (x, D, 3) follows $D = (x^2 - 1)/9$. If (x+9, D'', 3) is a solution then $D'' = ((x+9)^2 - 1)/9 = (x^2 + 81 +$
$+ 18x - 1)/9 = (x^2 - 1)/9 + (9 + 2x) =$. D + 9 + 2x.
Check: $(x + 9)^2 - 9D'' = (x + 9)^2 - 9D - 81 - 18x = x^2 + 81 + 18x - 9D - 81 - 18x = x^2 - x^2 + 1 = 1$ □
Example: If x = 33, $x^2 - 1 = 1.088$, which is not divisible by 9. Then (33; D, 3) is not a solution. If x = 35, $x^2 - 1 = 1.224$ and 1.224/9 = 136. From (35, 136, 3) follows x + 9 = 44 and D'' = D + 9 + 2x = 136 + 9 + 70 = 215.
Check: $44^2 - 215 \cdot 9 = 1936 - 1935 = 1$

Remark: Lemmas 9 and 10 show that solutions are associated to pairs in the relation (x, x + 2) and pairs are in the relation (x, x + 9). See Table 1, columns for y odd. This structure characterizes a sequence of solutions which grow without limit. The set of solutions is countably infinite. Moreover, there is not a Lemma similar to Lemma 5, since y is odd and, therefore, $y^2/2$ is not an integer.
Additionally notice that the first x-entries of the sequences of Table 1, show a non-monotone grow with respect to increasing values of y. This is due to the fact that the minimal solution obtained when y is odd –(Lemma 2)– is larger than the minimal solution obtained when y is even and larger than 2 –(Lemma 3)–.

Lemma 11: Let (x, D, 3) be a solution and D = 9d. Then (x, d, 3y) is also a solution.
Proof: From $x^2 - Dy^2 = 1$ follows $x^2 - 9dy^2 = x^2 - (d9)y^2 = x^2 - d(3y)^2 = 1$
Example: A solution with 9|D is (80, 711, 3), which leads to (80, 79, 9) since 79·9 = 711 and $80^2 - 79 \cdot 9^2 = 6400 - 6399 = 1$. □

Case y = 5 From $x^2 - Dy^2 = 1$ follows $x^2 - 25D = 1$ and $D = (x^2 - 1)/25$.

Lemma 12 Lemmas 9 and 10 may be extended to solutions with y = 5. The proof follows exactly the same steps as in the proofs of the Lemmas 9 and 10.
With Lemma 10, as in the case of y = 3, solutions are obtained in pairs. With Lemma 2, a first pair of solutions is (24, 23, 5), (26, 27, 5) followed by the pairs (49, 96, 5), (51, 104, 5); (74. 219, 5,), (76, 231, 5); (99, 392, 5), (101, 408, 5); …; (199, 1584, 5), (201, 1616, 5), obtained with Lemma 9. Notice that in each pair of this sequence, D increases by multiples of 4 = (y – 1). Moreover, except for the first solution of the first pair, x > D.
Taking in account that the first solution was obtained with x = 24, there is not a solution before (24, 23, 5).
The sequence of solutions grows without limit. The set of solutions is countably infinite.

Lemma 13: Let (x, D, 5) be a solution and D = 25d. Then (x, d, 5y) is also a solution.
Proof: From $x^2 - Dy^2 = 1$ follows $x^2 - 25dy^2 = x^2 - (d25)y^2 = x^2 - d(5y)^2 = 1$
Example: (1.249, 62.400, 5) = (1.249, (25·2.496), 5), is a solution with 25|D. This leads to a new solution with $y_{new} = 25$: (1.249, 2.496, 25), where $x^2 = 1.560.001$, d = 2.496, $(y_{new})^2 = 625$ and $d \cdot (y_{new})^2 = 2.496 \cdot 625 = 1.560.000$. Therefore, $x^2 - d(y_{new})^2 = 1$.

2.3 Special case: y is composite

Let (x, D, y) be a solution with $y = \prod_{i=1}^{n-1} y_i$ where for some $i, j \in \mathbb{Z}_n \backslash \{0\}$, y_i may equal y_j.
Consider the possibly simplest case (x, D, y) = (x, D, $y_1 \cdot y_2$). Then two new possibilities may be obtained based on (1):

$$x^2 - D(y_1 \cdot y_2)^2 = 1 \Rightarrow x^2 - (D(y_1)^2) \cdot (y_2)^2 = 1 \text{ and } x^2 - (D(y_2)^2) \cdot (y_1)^2 = 1 \quad (2)$$

The sequences of explicit solutions will share x-entries, however in different positions.
Examples: Let y = 12 = 3·4. Then from (x, D, y) two new solutions may be obtained: (x, 9D, y_2) and (x, 16D, y_1). The relationship among the solutions may be observed in Table 2. Similarly, let y = 15 = 3·5. Then also two new solutions are obtained: (x, 9D, y_2) and (x, 25D, y_1), as shown in Table 3.

Table 2: Sharing x-entries and scaling D-entries when $y = 12 = y_1 \cdot y_2 = 3 \cdot 4$

y = 12			$Y_1 = 3$			$y_2 = 4$		
i	x	D	*i*	x	16D	*i*	x	9D
1	71	35	15	71	560	17	71	315
2	73	37	16	73	592	18	73	333
3	143	142	31	143	2272	35	143	1278
4	145	146	32	145	2336	36	145	1314
5	215	321	48	215	5136	53	215	2889
6	217	327	49	217	5232	54	217	2943

Table 3: Sharing x-entries and scaling D-entries when $y = 15 = y_1 \cdot y_2 = 3 \cdot 5$

y = 15			$y_1 = 3$			$y_2 = 5$		
i	x	D	*i*	x	25D	*i*	x	9D
1	224	223	49	224	5575	17	224	2007
2	226	227	51	226	5625	18	226	2043
3	449	896	99	449	22400	35	449	8064
4	451	904	101	451	22450	36	451	8136
5	674	2019	149	674	50475	53	674	18171
6	676	2031	151	676	50525	54	676	18279

3. Closing Remarks

From the analysis of solutions for $y \in \{2, 3, 4, 5, 6, 7, 8\}$ in Table 1, sharing entries in Tables 2 and 3, and other examples, it may be extrapolated that for all $y \in \mathbb{N}$ there are solutions, that in each case it is possible to determine a minimal first solution and that there are simple rules that determine the growing structure of the sequences of solutions. Additionally, that the sequences of solutions have a growing structure leading to countably infinite sets of solutions. Moreover, in the case of y being composite, some solutions may be "inherited" from factor sequences, although locally generated with the growing rules. Since the union of countably infinite sets is countably infinite [10], then our results are consistent with this important known characteristic of the Pell Equation besides showing how to explicitly generate sequences of solutions.

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