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## On $\mathbb{Z}_p$ - linearity

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### Abstract

A recent result, valid in the context of certain topological  $\mathbb{Z}_p$  - modules vector spaces, is established in the context of topological  $\mathbb{Z}_p$

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### 1. Introduction

In the celebrated paper <sup>[2]</sup> of Banach it is implicit that, for real normed spaces  $E$  and  $F$  and for each continuous mapping  $u$  from  $E$  into  $F$ , the conditions

$$u(x + y) = u(x) + u(y) \text{ for all } x, y \in E \text{ (additivity)}$$

and

$$u(\lambda x + \mu y) = \lambda u(x) + \mu u(y) \text{ for all } \lambda, \mu \in \mathbb{R} \text{ and for all } x, y \in E \text{ (linearity)}$$

are equivalent. In <sup>[4]</sup>, the above-mentioned result has been extended to the context of topological vector spaces over certain topological fields. The purpose of the present note is to establish a similar result in the framework of topological  $\mathbb{Z}_p$  - modules.

### 2. Preliminaries and examples

For each prime natural number  $p$ , let  $\mathbb{Z}_p$  be the discrete valuation ring of  $p$  - adic integers <sup>[1, 5]</sup>;  $p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \dots \supset p^n\mathbb{Z}_p \supset p^{n+1}\mathbb{Z}_p \supset \dots$  is a fundamental system of neighborhoods of 0 in  $\mathbb{Z}_p$  consisting of ideals of  $\mathbb{Z}_p$ ,  $p\mathbb{Z}_p$  being a maximal ideal. A unitary  $\mathbb{Z}_p$  - module  $E$  endowed with a topology is a topological  $\mathbb{Z}_p$  - module <sup>[3, 6]</sup> if the mappings  $(x, y) \in E \times E \mapsto x + y \in E$  and  $(\lambda, x) \in \mathbb{Z}_p \times E \mapsto \lambda x \in E$  are continuous ( $E \times E$  and  $\mathbb{Z}_p \times E$  endowed with the corresponding product topology).

Let us mention a few examples of topological  $\mathbb{Z}_p$  - modules.

2.1 For each integer  $n \geq 1$ , the  $\mathbb{Z}_p$  - module  $(\mathbb{Z}_p)^n$ , endowed with the product topology, is a Hausdorff topological  $\mathbb{Z}_p$  - module.

2.2 The  $\mathbb{Z}_p$  - module  $(\mathbb{Z}_p)^\mathbb{N}$ , endowed with the product topology, is a Hausdorff topological  $\mathbb{Z}_p$  - module (obviously, 2.1 follows from 2.2).

2.3 For each non-empty set  $X$  and for each topological  $\mathbb{Z}_p$  - module  $F$ , the  $\mathbb{Z}_p$  - module  $\mathcal{F}(X; F)$  of all mappings from  $X$  into  $F$ , endowed with the topology of uniform convergence on the finite subsets of  $X$ , is a topological  $\mathbb{Z}_p$  - module, which is a Hausdorff space if  $F$  is a Hausdorff space (obviously, 2.2 follows from 2.3).

2.4 For each non-empty topological space  $X$  and for each topological  $\mathbb{Z}_p$ -module  $F$ , the  $\mathbb{Z}_p$ -module  $\mathcal{G}(X; F)$  of all continuous mappings from  $X$  into  $F$ , endowed with the topology of uniform convergence on the compact subsets of  $X$ , is a topological  $\mathbb{Z}_p$ -module, which is a Hausdorff space if  $F$  is a Hausdorff space.

2.5 For arbitrary topological  $\mathbb{Z}_p$ -modules  $E$  and  $F$ , the  $\mathbb{Z}_p$ -module  $\mathcal{L}(E; F)$  of all continuous  $\mathbb{Z}_p$ -linear mappings from  $E$  into  $F$ , endowed with the topology of uniform convergence on the bounded subsets of  $E$  [5, p. 114], is a topological  $\mathbb{Z}_p$ -module, which is a Hausdorff space if  $F$  is a Hausdorff space.

### 3. The result

We shall prove the following:

Proposition 3.1. Let  $E$  and  $F$  be the topological  $\mathbb{Z}_p$ -modules,  $F$  being a Hausdorff space. For a mapping  $u: E \rightarrow F$ , the following conditions are equivalent:

- $u(x + y) = u(x) + u(y)$  for all  $x, y \in E$ , and  $u$  is continuous at  $0 \in E$ ;
- $u(\lambda x + \mu y) = \lambda u(x) + \mu u(y)$  for all  $\lambda, \mu \in \mathbb{Z}_p$  and for all  $x, y \in E$ , and  $u$  is uniformly continuous.

Proof. It is obvious that (b) implies (a). Thus it remains to show that (a) implies (b). Indeed, since

$$u(x) - u(y) = u(x) + u(-y) = u(x - y)$$

for all  $x, y \in E$ , the continuity of  $u$  at the origin implies its uniform continuity.

Now, let  $\lambda \in \mathbb{Z}_p$  and  $x \in E$  be arbitrary. We claim that  $u(\lambda x) = \lambda u(x)$ . In fact, by the additivity of  $u$  and the Principle of Finite Induction,

$$u(nx) = nu(x)$$

for all  $n \in \mathbb{N}$ . Hence, if  $n \in \mathbb{Z}$  and  $n < 0$ ,

$$u(nx) = u(-((-n)x)) = -u((-n)x) = (-(-n))u(x) = nu(x).$$

Finally, by the density of  $\mathbb{Z}$  in  $\mathbb{Z}_p$  ([1], Proposition 1.2.3), there is a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  in  $\mathbb{Z}$  converging to  $\lambda$  in  $\mathbb{Z}_p$ . Since  $(\lambda_n x)_{n \in \mathbb{N}}$  converges to  $\lambda x$  in  $E$ , the continuity of  $u$  at  $\lambda x$  implies that  $(u(\lambda_n x))_{n \in \mathbb{N}}$  converges to  $u(\lambda x)$  in  $F$ . On the other hand,  $u(\lambda_n x) = \lambda_n u(x)$  for all  $n \in \mathbb{N}$ , and therefore  $(u(\lambda_n x))_{n \in \mathbb{N}}$  converges to  $\lambda u(x)$  in  $F$ . Consequently the assumption that  $F$  is a Hausdorff space furnishes  $u(\lambda x) = \lambda u(x)$ , which concludes the proof.

### 4. Conclusion

In this short communication it is shown that, under quite general assumptions,  $\mathbb{Z}_p$ -linearity comes from additivity.

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