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On *Zp* - linearity

[1] Pombo Jr Dinamérico P, [2] Farias Marcelo F, [3] Mauro Patricia CG
[1] Instituto de Matemática e Estatística da UFF, Brazil
[2] Instituto Multidisciplinar da UFRRJ, Brazil
[3] Centro Interdisciplinar de Ciências da Natureza da UNILA, Brazil

Corresponding Author: **Pombo Jr Dinamérico P**

Abstract
A recent result, valid in the context of certain topological vector spaces, is established in the framework of topological $\mathbb{Z}_p$ - modules.

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1. Introduction

In the celebrated paper [2] of Banach it is implicit that, for real normed spaces *E* and *F* and for each continuous mapping *u* from *E* into *F*, the conditions:

$$u(x+y) = u(x) + u(y) \text{ for all } x, y \in E \text{ (additivity)}$$

and,

$$u(\lambda x + \mu y) = \lambda u(x) + \mu u(y) \text{ for all } \lambda, \mu \in \mathbb{R} \text{ and for all } x, y \in E \text{ (linearity)}$$

are equivalent. In [4], the above-mentioned result has been extended to the context of topological vector spaces over certain topological fields. The purpose of the present note is to establish a similar result in the framework of topological $\mathbb{Z}_p$ - modules.

2. Preliminaries and examples

For each prime natural number *p*, let $\mathbb{Z}_p$ be the discrete valuation ring of *p* - adic integers [1, 5]; $p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \cdots \supset p^n\mathbb{Z}_p \supset p^{n+1}\mathbb{Z}_p \supset \cdots$ is a fundamental system of neighborhoods of 0 in $\mathbb{Z}_p$ consisting of ideals of $\mathbb{Z}_p$, $p\mathbb{Z}_p$ being a maximal ideal. A unitary $\mathbb{Z}_p$ - module *E* endowed with a topology is a topological $\mathbb{Z}_p$ - module [3, 6] if the mappings $(x,y) \in E \times E \mapsto x + y \in E$ and $(\lambda, x) \in \mathbb{Z}p \times E \mapsto \lambda x \in E$ are continuous (*E* x *E* and $\mathbb{Z}p \times E$ endowed with the corresponding product topology).

Let us mention a few examples of topological $\mathbb{Z}_p$ - modules.

2.1 For each integer $n \geq 1$, the $\mathbb{Z}_p$ - module $(\mathbb{Z}p)^n$, endowed with the product topology, is a Hausdorff topological $\mathbb{Z}_p$ - module.

2.2 The $\mathbb{Z}_p$ - module $(\mathbb{Z}p)^n$, endowed with the product topology, is a Hausdorff topological $\mathbb{Z}_p$ - module (obviously, 2.1 follows from 2.2).

2.3 For each non-empty set *X* and for each topological $\mathbb{Z}_p$ - module *F*, the $\mathbb{Z}_p$ - module $\mathcal{F}(X;F)$ of all mappings from *X* into *F*, endowed with the topology of uniform convergence on the finite subsets of *X*, is a topological $\mathbb{Z}_p$ - module, which is a Hausdorff space if *F* is a Hausdorff space (obviously, 2.2 follows from 2.3).

2.4 For each non-empty topological space X and for each topological $\mathbb{Z}_p$ - module F, the $\mathbb{Z}_p$ - module $\mathcal{C}(X;F)$ of all continuous mappings from X into F, endowed with the topology of uniform convergence on the compact subsets of X, is a topological $\mathbb{Z}_p$ - module, which is a Hausdorff space if F is a Hausdorff space.
2.5 For arbitrary topological $\mathbb{Z}_p$ - modules E and F, the $\mathbb{Z}_p$ - module $\mathcal{L}(E;F)$ of all continuous $\mathbb{Z}_p$ - linear mappings from E into F, endowed with the topology of uniform convergence on the bounded subsets of E [5, p. 114], is a topological $\mathbb{Z}_p$ - module, which is a Hausdorff space if F is a Hausdorff Space.

3. The result

We shall prove the following:
Proposition 3.1. Let E and F be the topological $\mathbb{Z}_p$ - modules, F being a Hausdorff space. For a mapping $u: E \to F$, the following conditions are equivalent:

a. $u(x+y) = u(x) + u(y)$ for all $x, y \in E$, and u is continuous at $0 \in E$;
b. $u(\lambda x + \mu y) = \lambda u(x) + \mu u(y)$ for all $\lambda, \mu \in \mathbb{Z}_p$ and for all $x, y \in E$, and u is uniformly continuous.

Proof. It is obvious that (b) implies (a). Thus it remains to show that (a) implies (b). Indeed, since:

$$u(x) - u(y) = u(x) + u(-y) = u(x-y)$$

For all $x, y \in E$, the continuity of u at the origin implies its uniform continuity.

Now, let $\lambda \in \mathbb{Z}_p$ and $x \in E$ be arbitrary. We claim that $u(\lambda x) = \lambda u(x)$. In fact, by the additivity of u and the Principle of Finite Induction,

$$u(nx) = nu(x)$$

For all $n \in \mathbb{N}$. Hence, if $n \in \mathbb{Z}$ and $n < 0$,

$$u(nx) = u\left(-((-n)x)\right) = -u((-n)x) = \left(-(-n)\right)u(x) = nu(x).$$

Finally, by the density of $\mathbb{Z}$ in $\mathbb{Z}_p$ ([1], Proposition 1.2.3), there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ converging to λ in $\mathbb{Z}_p$. Since $(\lambda_n x)_{n\in\mathbb{N}}$ converges to λx in E, the continuity of u at λx implies that $(u(\lambda_n x))_{n\in\mathbb{N}}$ converges to $u(\lambda x)$ in F. On the other hand, $u(\lambda_n x) = \lambda_n u(x)$ for all $n \in \mathbb{N}$, and therefore $(u(\lambda_n x))_{n\in\mathbb{N}}$ converges to $\lambda u(x)$ in F. Consequently the assumption that F is a Hausdorff space furnishes $u(\lambda x) = \lambda u(x)$, which concludes the proof.

4. Conclusion

In this short communication it is shown that, under quite general assumptions, $\mathbb{Z}_p$ - linearity comes from additivity.

5. References

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