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First – Order Differential Equations and Some Applications Using MATLAB

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Abstract
This study introduced real life application of first order differential equation. In this study We basically discussed about different types of differential equation and the solution of first order differential equation and application of first order differential equation in different field of science and technology using MALAB. Further, Newton's law of cooling and orthogonal trajectory has been incorporated. Study about convective boundary condition and it is used for increasing the temperature. The study followed applied analytical approach. The study reached the most important results, If f and $\frac{\partial f}{\partial y}$ are continuous near the initial point, there is a unique solution in a small interval around x_0, first-order ODEs yield specific mathematical patterns when applied to physical systems, a grid of small slopes that visually represent the "result" of the ODE without solving it algebraically. We recommend that researchers conduct further research on solutions to these differential equations using MATLAB and other software to solve and plot them.

Keywords: MATLAB, first-order ODEs, Sudan

Introduction

The equation contains one or more than one derivative is known as differential equations or in other words we can say the equation that contain derivative of one or more dependent variable with respect to one or more independent variables. Differential equations have remarkable ability to predict the world around us [1]. In mathematics a first ordinary differential equation is an equation that contain only first derivative and it has many applications in mathematics, biology, physics engineering and other subjects. Because any physical laws and relations appear mathematically in the form of such equations [2]. The study of dynamics, a fundamental branch of classical physics, is primarily concerned with understanding the motion of objects and the various forces that influence that motion. A significant portion of this field relies on the use of mathematical tools to model and predict physical behavior over time [3]. Many mathematical models in science and engineering fields can be formulated in the form linear and nonlinear ordinary differential equations which need an analytical method to solve the exact equations [4]. A first –order differential equation is an equation that contain only first derivative, and it has many applications in mathematics, physics, engineering and many other subjects. In this study, we discussed the applications of the Population growth and decay, and Mixing problems [5]. In these notes we present an introductory review of the properties and solutions of 1st order ordinary differential equations. we introduce the concept of 1-parameter family of level curves which we show forms the building block for understanding the solution properties of 1st order ODEs. We then review some basic definitions regarding ordinary differential equations, and then proceed with the main discussion on several methods for solving ODEs [6].

Differential Equation 1: A differential equation is a mathematical equation that relates a function with its derivatives of one or more dependent variables with respect to one or more independent variables [3].
Now consider the following system of polynomial 1ODEs in the plane:

$$\dot{x} = f(x.y)$$
$$\dot{y} = g(x.y) \tag{1}$$

Where f and g are coprime polynomial in $(x.y)$ and the dot means derivative with respect to a parameter t ($\dot{x} \equiv du/dt$).

Definition 2: A function $I(x.y)$ is called a first integral of the system (1) if I is constant over the solutions of [8].

Theorem 1: (Prelle –Singer). If the system (1) presents an elementary first integral I, then there exists an integrating factor R for the system (1) of the form $R = \prod_i p_i^{n_i}$, where the p_i are irreducible Darboux polynomials of system (1) and n_i are rational numbers.

Proof:

Since $\frac{X(R)}{R} = -\mathbf{div}(X) = -(f_x + g_y)$ Substituting $R = \prod_i p_i^{n_i}$ we obtain (See (1));

$$\sum_i \frac{X(p_i)}{p_i} = \sum_i n_i q_i = -(f_x + g_y) \tag{2}$$

Where the polynomial $q_i\left(=\frac{X(p_i)}{p_i}\right)$ are called cofactors of the Darboux polynomials p_i. Therefore, a possible method to find an elementary first integral is:

Prelle-Singer method:(sketch)

- Determine the DPs p_i associated with the system.
- Find numbers n_i that satisfy $\sum_i n_i q_i = -(f_x + g_y)$.
- Construct the integrating factor $R = \prod_i p_i^{n_i}$ and find a first integral $I(x.y)$ by quadratures [9].

We consider the existence and uniqueness of solutions $y = y(x)$ of the differential equation:

$$y' = f(x.y). \tag{3}$$

Satisfying,

$$y(a) = c. \tag{4}$$

Where a is a point in the domain of y and c is (another) constant. In order to achieve our aim, we must place restrictions on the function f:

(a) f is continuous in a region U of the $(x.y)$ -plane which contains the rectangle:

$$R = \{(x, y): |x - a| \leq h, |y - c| \leq k\}$$

where h and k are positive constants,

(b) f satisfies the following 'Lipschitz condition' for all pairs of points (x, y_1), (x, y_2) of U:

$$|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|,$$

Where A is a (fixed) positive constant.

Restriction (a) implies that f must be bounded on R. Letting:

$$M = \sup\{|f(x, y)|: (x, y) \in R\},$$

We add just one further restriction, to ensure, as we shall see, that the functions to be introduced are well defined:

(c) $Mh \leq k$.

We would also make a remark about the ubiquity of restriction (b). Such a Lipschitz condition must always occur when the partial derivative $\frac{\partial f}{\partial y}$ exists as a bounded function on U: if a bound on its modulus is $P > 0$. we can use the Mean Value Theorem of the differential calculus as applied to $f(x.y)$ considered as a function of y alone, to write, for some y_0 between y_1 and y_2:

$$|f(x.y_1) - f(x.y_2)| = \left|\frac{\partial f}{\partial y}(x.y_0)\right| |y_1 - y_2| \leq P|y_1 - y_2|$$

Theorem 2: (Cauchy–Picard) When the restrictions (a), (b), (c) are applied, there exists, for $|x - a| \leq h$ a solution to the problem consisting of the differential equation (1) is unique amongst functions with graphs lying in U.

Proof: We apply Picard's method and define a sequence (y_n) of functions:

$y_n: [a - h, a + h] \to \mathbb{R}$ by the iteration:

$$y_0(x) = c$$

$$y_n(x) = c + \int_a^x f\left(t. y_{n-1}(t)\right)dt. \qquad (n \geq 1)$$

As f is continuous, $f(t, y_{n-1}(t))$ is a continuous function of t whenever $y_{n-1}(t)$ is the iteration defines a sequence (yn) of continuous functions on $[a-h.a+h]$, provided that $f(t, y_{n-1}(t))$ is defined on $[a-h.a+h]$; that is, provided that:

$|y_n(x) - c| \leq k$, for all $x \in [a-h, a+h]$ and $n = 1, 2,...$

To see that this is true, we work by induction. Clearly,

$|y_0(x) - c| \leq k$, for each $x \in [a-h, a+h]$.

If $|y_{n-1}(x) - c| \leq$ k for all $x \in [a-h, a+h]$, where n ≥ 1, then $f(t, y_{n-1}(t))$ is defined on $[a-h, a+h]$ and, for x in this interval,

$$|y_n(x) - c| = \left|\int_a^x f\left(t. y_{n-1}(t)\right)dt\right| \leq M|x-a| \leq Mh \leq k. \qquad (n \geq 1)$$

The induction is complete [7].
We provide next the inductive step of the proof of:

$|y_n(x) - y_{n-1}(x)| \leq \frac{A^{n-1}M}{n!}|x-a|^n$, for all $x \in [a-h, a+h]$ and all $n \geq 1$.

Suppose that:

$|y_{n-1}(x) - y_{n-2}(x)| \leq \frac{A^{n-2}M}{(n-1)!}|x-a|^{n-1}$, for all $x \in [a-h, a+h]$ and all $n \geq 2$.

Then, using the Lipschitz condition (b),

$$|y_n(x) - y_{n-1}(x)| = \left|\int_a^x \left(f\left(t. y_{n-1}(t)\right) - f\left(t. y_{n-2}(t)\right)\right)dt\right|$$

$$\leq \left|\int_a^x \left(f\left(t. y_{n-1}(t)\right) - f\left(t. y_{n-2}(t)\right)\right)dt\right| \qquad (5)$$

$$\leq \left|\int_a^x A|y_{n-1}(t) - y_{n-2}(t)|dt\right|$$

$$\leq A \cdot \frac{A^{n-2}M}{(n-1)!}\left|\int_a^x |t-a|^{n-1}dt\right|$$

$$= \frac{A^{n-1}M}{n!}|x-a|^n. \qquad (n \geq 2) \qquad (6)$$

for every x in $[a-h, a+h]$.
Note: The reader may wonder why we have kept the outer modulus signs in (5) above. The reason is that it is possible for x to be less than a, while remaining in the interval $[a-h.a+h]$.
Putting

$S = f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)), \qquad (n \geq 2)$

is actually being applied as follows when $x < a$:

$$\left|\int_a^x Sdt\right| = \left|-\int_s^a Sdt\right| = \left|\int_x^a Sdt\right| \leq \int_x^a |S|dt = -\int_a^x |S|dt = \left|\int_a^x |S|dt\right|$$

Similarly, for $x < a$,

$$\left|\int_a^x |t-a|^{n-1}dt\right| = \left|-\int_x^a (a-t)^{n-1}dt\right| = \frac{(a-x)^n}{n} = \frac{|x-a|^n}{n}$$

Establishing (6).
Continuing with the proof and putting, for each $n \geq 1$,

$$M_n = \frac{A^{n-1}Mh^n}{n!}$$

We see that we have shown that $|y_n(x) - y_{n-1}(x)| \le M_n$ for all $n \geq 1$ and all x in $[a-h.a+h]$ However,

$$\sum_{n=1}^{\infty} M_n$$

of constants, converging to

$$\frac{M}{A}\,(e^{Ah}-1)$$

So, the Weierstrass M-test may again be applied to deduce that the series:

$$\sum_{n=1}^{\infty}(y_n - y_{n-1})$$

Converges uniformly on $[a-h.a+h]$ Hence, the sequence (y_n) converges uniformly to, say, y on $[a-h.a+h]$ As each y_n is continuous (see above).
Further, y_n (t) belongs to the closed interval $[c-k.c+k]$ for each n and each t $\in [a-h.a+h]$. Hence, $y(t) \in [c-k.c+k]$ for each $t \in [a-h.a+h]$ and $f(t.y(t))$ is a well-defined continuous function on $[a-h.a+h]$. Using the Lipschitz condition, we see that:

$$|f(t.y(t)) - f(t.y_n(t))| \le A|y(t) - y_n(t)| \qquad (n \ge 0)$$

For each t in $[a-h.a+h]$; so, the sequence $f(t.y_n(t))$ converges uniformly to $f(t.y(t))$ on $[a-h.a+h]$.

$$\int_a^x f(t.y_{n-1}(t))dt \quad \to \quad \int_a^x f(t.y(t))dt$$

As $n \to \infty$, So, letting $n \to \infty$ in the equation:

$$y_n(x) = c + \int_a^x f(t.y_{n-1}(t))dt$$

Defining our iteration, we obtain:

$$y(x) = c + \int_a^x f(t.y(t))dt \quad (7)$$

Note that $y(a) = c$. As the integrand in the right-hand side is continuous, we may differentiate with respect to x to obtain"

$$y'(x) = f\,(x.y(x))$$

Thus $y = y(x)$ satisfies the differential equation (1). We have shown that there exists a solution to the problem.
The uniqueness of the solution again follows the pattern of our work in section 1.2. If $y = Y(x)$ is a second solution of (1) with graph lying in U, then as $y(x)$ and $Y(x)$ are both continuous functions on the closed and bounded interval $[a-h.a+h]$, there must be a constant N such that:

$$|y(x) - Y(x)| \le N \text{ for all } x \in [a-h.a+h]$$

Integrating $Y'(t) = f\,(t.Y(t))$ with respect to t, from a to x, we obtain:

$$Y(x) = c + \int_a^x f(t.Y(t))dt$$

Since $Y(a) = c$. So, using (7) and the Lipschitz condition made available to us by the graph of $y = Y(x)$ lying in U,

$$|y(x) - Y(x)| = \left|\int_a^x \big(f(t.y(t)) - f(t.Y(t))\big)\,dt.\right|$$
$$\le \left|\int_a^x A\,|y(t) - Y(t)|dt\right| \qquad (8)$$

$\leq A\,N\,|x-a|$. for all $t \in [a-h.a+h]$

We leave it to the reader to show by induction that, for every integer n and every:

$$x \in [a - h.a + h]$$

$$|y(x) - Y(x)| \leq \frac{A^n N}{n!} |x-a|^n$$

As the right-hand side of this inequality may be made arbitrarily small, $y(x) = Y(x)$ for each x in $[a - h.a + h]$. Our solution is thus unique.

Note (a) For continuous y, the differential equation (1) is equivalent to the integral equation (7).

(b) The analysis is simplified and condition (c) omitted if f is bounded and satisfies the Lipschitz condition in the strip:

$$\{(x.y) : a-h \leq x \leq a+h\}$$

(c) Notice that if the domain of f were sufficiently large and $\frac{\partial f}{\partial y}$ were to exist and be bounded there, then our work in the paragraph prior to the statement of the theorem would allow us to dispense with the graph condition for uniqueness.

Two simultaneous equations

It should be said at the outset that the methods of this section can be applied directly to the case of any finite number of simultaneous equations. The methods involve a straightforward extension of those employed in the last section and, for this reason, many of the details will be left for the reader to fill in.

We now seek solutions $y = y(x).\ z = z(x)$ of the simultaneous differential equations:

$$y' = f(x.y.z). \quad z' = g(x.y.z) \tag{9}$$

Which satisfy,

$$y(a) = c\,. \quad z(a) = d \tag{10}$$

Where a is a point in the domains of y and z, c and d are also constants, and where

(d) f and g are continuous in a region V of $(x.y.z)$-space which contains the cuboid:

$$S = \{(x.y.z) : |x-a| \leq h. \quad \max(|y-c|.|z-d|) \leq k\}$$

Where $h.k$ are non-negative constants,

(e) f and g satisfy the following Lipschitz conditions at all points of V:

$$|f(x.y_1.z_1) - f(x.y_2.z_2)| \leq A \max(|y_1 - y_2|.|z_1 - z_2|)$$

$$|g(x.y_1.z_1) - g(x.y_2.z_2)| \leq B \max(|y_1 - y_2|.|z_1 - z_2|)$$

Where A and B are positive constants,

(f) $\max(M.N)\,.\ h \leq k$

Where $M = \sup\{|f(x.y.z)| : (x.y.z) \in S\}$ and $N = \sup\{|g(x.y.z)| : (x.y.z) \in S\}$

It is convenient (especially in the n-dimensional extension!) to employ the vector notation:

$$y = (y.z). \quad f = (f.g). \quad c = (c.d). \quad A = (A.B). \quad M = (M.N)$$

The reader can then easily check that, with use of the 'vector norm',

$$|y| = \max(|y|\,.|z|)$$

Where $y = (y.z)$. the above problem reduces to:

$$y' = f(x.y) \tag{9}$$

Satisfying,

$$y(a) = c \quad (10)$$

Where;
(d') f is continuous in region V containing:

$$S = \{(x.y) \ : \ |x - a| \ \leq h. \quad |y - c| \leq k\}$$

(e') f satisfies the Lipschitz condition at all points of V:

$$|f(x.y_1) - f(x.y_2)| \leq |A|\,|y_1 - y_2|)$$

and

$(f')\,|M|\,h \ \leq k$

The existence of a unique solution to (9') subject to (10') can now be demonstrated by employing the methods of the first – order differential equation in a single independent variable to the iteration.

$$y_0(x) = c.$$

$$y_n(x) = c + \int_a^x f\big(t.y_{n-1}(t)\big)dt. \qquad (n \geq 1) \quad (11)$$

We thus have the following extension of Theorem (2).

Example 1: Mathematica experiments with level curves
Suppose the family of level curves is given by:

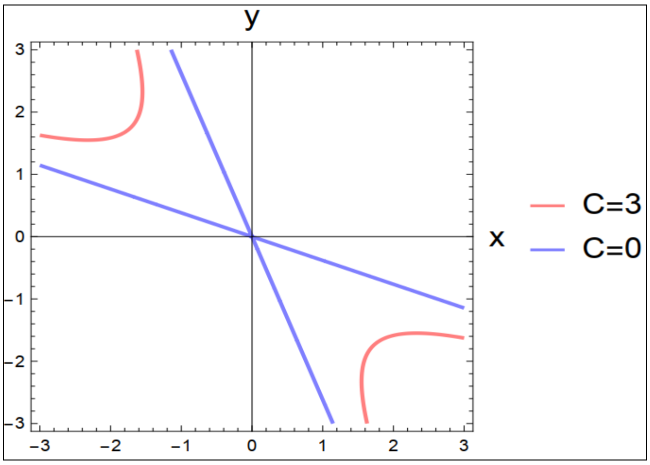
$$g(x.y) = \ y^2 + 3xy + x^2 = c$$

Display the level curves for $c = \{-3.-2.-1.\ 0\ .\ 1\ .\ 2\ .\ 3\ \}$

Solution:
For each value of c we need to find the $(x\,.\,y)$ pair that satisfies:

$$g(x.y) = \ y^2 + 3xy + x^2 = c \quad (12)$$

Since $g(x.y) = c$ defines a surface of height c. we need to determine a contour plot of this function. This can be readily done in Mathematica using the **Contour Plot** function. Here is code that achieves the desired result.



MATLAB solution to display the level curves of:

$$g(x.y) = \ y^2 + 3xy + x^2 = c. \qquad c \in \{-3.-2.-1.\ 0\ .\ 1\ .\ 2\ .\ 3\ \}$$ [6].

MATLAB Code (Level Curves / Contour Plot)

```
clear all; clc
% Define the grid
[x,y] = meshgrid(linspace(-5,5,400), linspace(-5,5,400));
% Define the function
g = x.^2 + 3*x.*y + y.^2;
% Define the level values
c = [-3 -2 -1 0 1 2 3];
% Plot level curves
figure
contour(x, y, g, c, 'LineWidth', 2)
grid on
axis equal
xlabel('x')
ylabel('y')
title('Level Curves of g(x,y) = x^2 + 3xy + y^2')
% Label contours
clabel(contour(x, y, g, c))
```

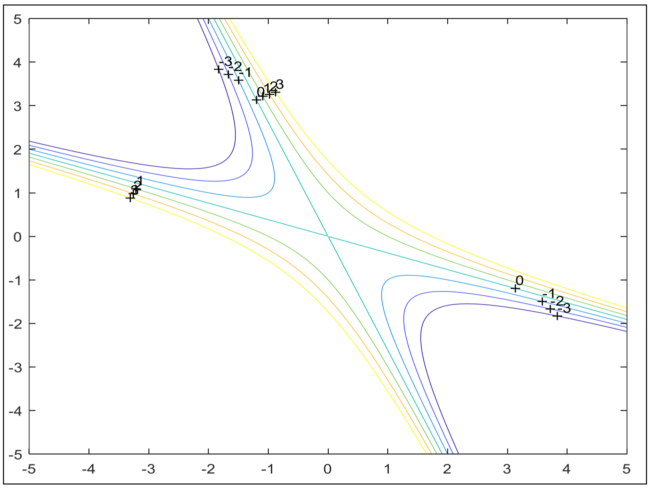


Fig 1: Level Curves of $g(x.y) = y^2 + 3xy + x^2$

What This Produces

Each curve represents the set $y^2 + 3xy + x^2$. The curves are conic sections (hyperbolas and degenerate cases). axis equal ensures correct geometric interpretation. clabel adds numerical labels to each level curve.

Mathematical Insight (Brief)

The quadratic form matrix is:

$$A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 1 \end{bmatrix}$$

With determinant $1 - \frac{9}{4} < 0$, hence the level sets are hyperbolas for $c \neq 0$, and intersecting lines for $c = 0$.

3D surface with level curves
MATLAB solution that visualizes

$$g(x.y) = y^2 + 3xy + x^2$$

As a 3D surface and shows the level curves corresponding to:

$$c = \{-3.-2.-1.0.1.2.3\}$$

Mathematical setup (brief)

The function

$$g(x.y) = y^2 + 3xy + x^2 = c$$

is a quadratic form. Its level sets $g(x.y) = c$ are conic sections (hyperbolas for negative c, degenerate at $c = 0$, ellipses for positive c).

MATLAB code — 3D surface with level curves

```
clear all; clc
% Define the domain
x = linspace(-5,5,400);
y = linspace(-5,5,400);
[X,Y] = meshgrid(x,y);
% Define the function
G = X.^2 + 3.*X.*Y + Y.^2;
% 3D surface plot
figure
surf(X,Y,G,'EdgeColor','none')
colormap parula
colorbar
hold on
% Level curve values
c = [-3 -2 -1 0 1 2 3];
% Overlay level curves on the surface
contour3(X,Y,G,c,'k','LineWidth',2)
% Labels and title
xlabel('x')
ylabel('y')
zlabel('g(x,y)')
title('3D Surface and Level Curves of g(x,y)=x^2+3xy+y^2')
view(45,30)
grid on
```

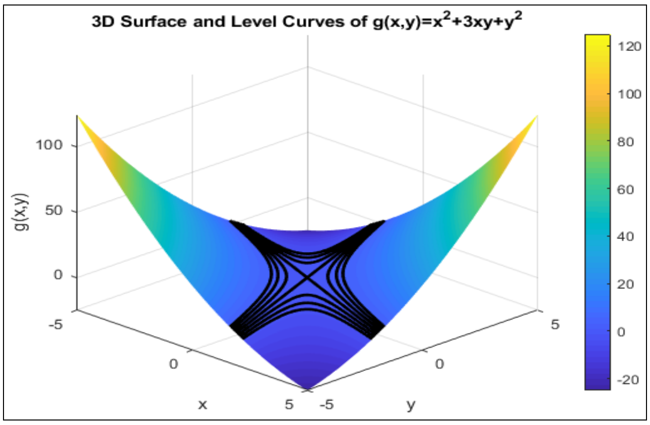


Fig 2: 3D Surface and Level Curves of $g(x.y) = y^2 + 3xy + x^2$

What this plot shows (interpretation)

The surface represents $z = g(x.y)$. The black contour lines are intersections of the surface with planes $z = c$. Negative c: open curves (hyperbolic behavior). $c = 0$: degenerate case. Positive c: Closed curves.

Example 2: More experiments with level curves

Consider the family of level curves given by:

$$g(x.y) = y^2 - 2y - x^3 - 2x = c \tag{13}$$

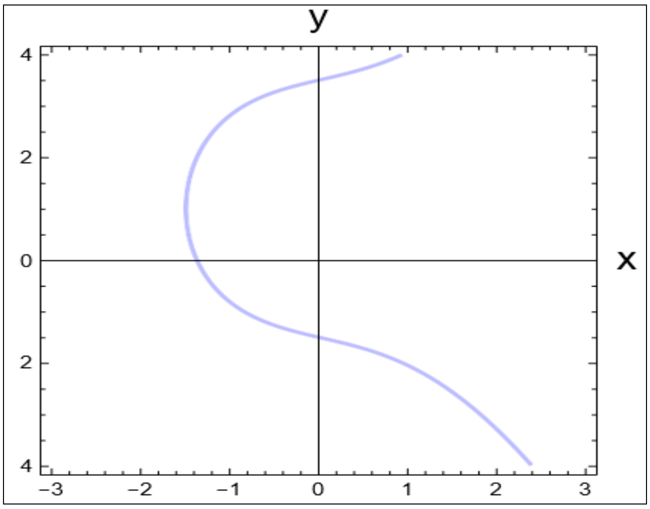
Determine the value of c such that a curve passes through the point $x = 0. \quad y = -1.5$. Plot the level curve and show its relationship to the family.

Solution

For $x = 0. \quad y = -1.5$ we have:

$$g(0. -1.5) = (-1.5)^2 - 2(-1.5) = c \tag{14}$$

Thus $c = 5.25$ [6].



MATLAB Solution
We are given the family of level curves:

$$g(x.y) = y^2 - 2y - x^3 - 2x = c$$

1. Determine the value of c
A curve passes through the point:

$$(x.y) = (0. -1.5)$$

Substitute into $g(x.y) = c$

$$c = (-1.5)^2 - 2(-1.5) - 0^3 - 2(0)$$

$$= 2.25 + 3$$

$$= 5.25$$

$$c = 5.25$$

2. MATLAB Plot: Family of Level Curves + Highlighted Curve
The following MATLAB code:
Plots several level curves (the family). Highlights the specific curve c = 5.25. Marks the point (0,−1.5).

MATLAB code Family of Level Curves + Highlighted Curve

```
clear all
clc
% Define grid
[x,y] = meshgrid(linspace(-3,3,400), linspace(-4,4,400));
% Define the function
g = y.^2 - 2*y - x.^3 - 2*x;
% Family of level values
c_family = [-10 -5 0 2 5 7 10];
% Specific value of c
c_special = 5.25;
figure
hold on
% Plot family of level curves
contour(x, y, g, c_family, 'LineWidth', 1.5)
% Plot the special level curve
contour(x, y, g, [c_special c_special], ...
    'LineWidth', 3, 'LineColor', 'r')
```

```
% Mark the given point
plot(0, -1.5, 'ko', 'MarkerFaceColor', 'k')
grid on
axis equal
xlabel('x')
ylabel('y')
title('Level Curves of g(x,y) = y^2 - 2y - x^3 - 2x')
legend('Family of level curves', 'c = 5.25', 'Point (0, -1.5)', ...
    'Location', 'best')
hold off
```

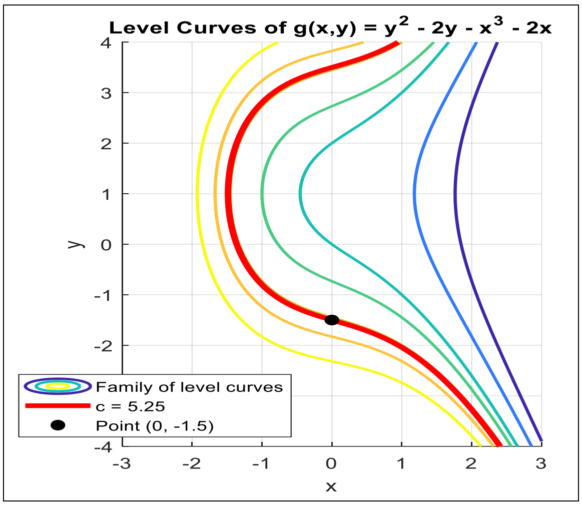


Fig 3: Level Curves of $g(x.y) = y^2 - 2y - x^3 - 2x$

3. Interpretation

The level curves are nonlinear and asymmetric due to the cubic term x^3. For fixed x, the curves are parabolic in y. The red curve (c=5.25) clearly belongs to the family and passes exactly through (0,−1.5). This confirms both the analytical result and the geometric interpretation.

3D surface with level curves

1) Determine the value of c

The function is:

$$g(x.y) = y^2 - 2y - x^3 - 2x = c$$

We are told that a level curve passes through the point:

$$(x.y) = (0.-1.5)$$

Substitute into $g(x.y) = c$,

$$c = (-1.5)^2 - 2(-1.5) - 0^3 - 2(0)$$

$$= 2.25 + 3$$

$$= 5.25$$

$$c = 5.25$$

So the required level curve is:

$$y^2 - 2y - x^3 - 2x = 2.25$$

2) MATLAB code - 3D surface with level curves

This plot shows:

the surface $z = g(x.y)$, the family of level curves, and highlights the specific level curve passing through $(0.-1.5)$.

MATLAB code — 3D surface with level curves

```
clear all;
clc
% Define domain
x = linspace(-3,3,400);
y = linspace(-4,4,400);
[X,Y] = meshgrid(x,y);
% Define the function
G = Y.^2 - 2.*Y - X.^3 - 2.*X;
% Value of c at the given point
c0 = 5.25;
% 3D surface plot
figure
surf(X,Y,G,'EdgeColor','none')
colormap turbo
colorbar
hold on
% Family of level curves
c = -10:2:10;
contour3(X,Y,G,c,'k','LineWidth',1)
% Highlight the required level curve
contour3(X,Y,G,[c0 c0],'r','LineWidth',3)
% Mark the given point
plot3(0,-1.5,c0,'ro','MarkerFaceColor','r','MarkerSize',8)
% Labels and title
xlabel('x')
ylabel('y')
zlabel('g(x,y)')
title('3D Surface and Level Curves of g(x,y)=y^2-2y-x^3-2x')
view(45,30)
grid on
```

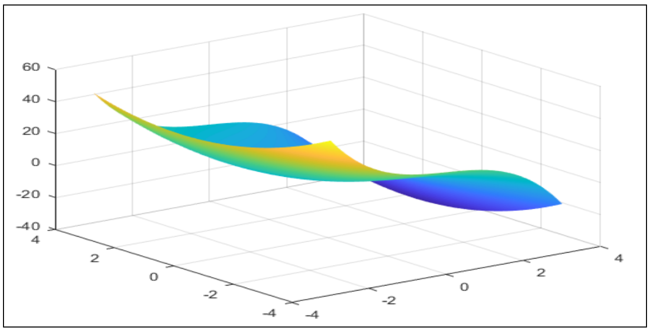


Fig 4: 3D Surface and Level Curves of g(x,y)=$y^2 - 2y - x^3 - 2x$

3) Interpretation (what the plot shows)

The surface represents

$$z = y^2 - 2y - x^3 - 2x$$

Each contour corresponds to a plane $z = c$. The red curve is the specific level curve $c = 5.25$. The red point confirms that the curve passes through $(0. \; - 1.5. \;\; 2.25)$. The asymmetry of the surface reflects the cubic term x^3, which causes strong directional bending of the level curves.

Solution of First Order Linear ODE:

First order linear ODEs are solved by using integrating factor method. At first we write the equation, then convert the equation in standard form, then find the integrating factor of the given equation. In the next step multiply the equation by the integrating factor on the both sides, then integrate the whole equation with respect to the independent variable present in the equation. Now by solving the dependent variable, we got the required solution of the given first order linear ODE.

e the standard form of the first order linear ODE is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Where, y is the dependent variable and x is the independent variable.
Now we find integrating factor $u(x)$.

$$u(x) = e^{\int p(x)dx}$$

So the solution is, multiply the integrating factor with the standard form of the given equation on the both sides.

$$\therefore u(x)\{\frac{dy}{dx} + P(x)y\} = u(x).Q(x)$$

$$\Rightarrow u(x)\frac{dy}{dx} + u(x)\cdot P(x)y = u(x)\cdot Q(x)$$

$$\Rightarrow u(x)dy + u(x)P(x)ydx = u(x)\cdot Q(x)dx$$

Now integrating both sides $w.r.t\ x$, we get:

$$\Rightarrow \int u(x)\, dy + \int u(x) \cdot P(x)\cdot ydx = \int u(x) \cdot Q(x)dx$$

Then solve for y to got the solution of the given differential equation.

Example 3: A 160 pound (total weight) skydiver falls with air resistance of $\frac{1}{3}v$ pounds. Find the velocity after 4 seconds of free-fall, when the parachute opens.

Solution:

Given that, weight (downward)(+ve) W = 160 lb and air resistance (upward)(-ve) = $\frac{1}{3}v$

According to Newton's Second Law: $F = ma$, where the net force $F = F_1 + F_2$, mass (m), acceleration $(a) = \frac{dv}{dt}$, weight (W), mg or $m = \frac{W}{g}$

(acceleration due to gravity g = 32 ft/s² (approx)).

Substituting the mass m = 5 slugs

Now, the net force, $F = 160 - \frac{1}{3}v$

$$\Rightarrow ma = 160 - \frac{1}{3}v$$

$$\Rightarrow 5\frac{dv}{dt} = 160 - \frac{1}{3}v$$

$$\Rightarrow \frac{dv}{dt} = 40 - \frac{1}{15}v$$

Here we solve the differential equation by separation of variable, we have:

$$\Rightarrow \frac{dv}{40 - \frac{1}{15}v} = dt$$

By putting integration on both sides, we have,

$$\Rightarrow \int \frac{dv}{40 - \frac{1}{15}v} = \int dt$$

$$\Rightarrow 15\int \frac{dv}{480 - v} = \int dt$$

$$\Rightarrow \int \frac{dv}{v - 480} = -\frac{1}{15}\int dt$$

$$\Rightarrow \ ln(v - 480) = -\frac{1}{15} + \ C_0$$

Where C_0 is the integrating constant:

$$\Rightarrow v \ = 480 + C \ e^{-\frac{1}{15}+ \ C_0}$$

Where C = $e^{\ C_0}$

By applying initial condition, that at $t = 0$ and $v = 0$ (falls from rest):

$$\therefore \ \ 0 = 480 - C \ e^0$$

$$\Rightarrow C - 480$$

Here the velocity function,

$$\Rightarrow v(t) = 480 + C \ e^{-\frac{1}{15}t}$$

$$= 480 - 480 \ e^{-\frac{1}{15}t}$$

$$= 480(1 - \ e^{-\frac{1}{15}t})$$

Now we have to find velocity at t = 4 seconds,

$$\therefore \ \ v(t) = 480(1 - \ e^{-\frac{1}{15}\times 4})$$

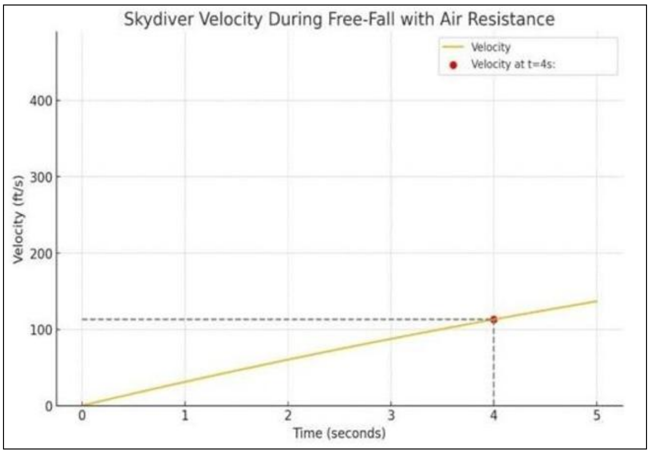
$$= 480(1 - \ e^{-\frac{4}{15}})$$

≈ 480 (1 - 0.765992833836)

≈ 480 (0.23407166164)

≈ 112.3543 ft/s

∴ So, the velocity is 112.3543 ft/s when parachute opens [1].



Types of Differential Equation

1. Ordinary differential equations
2. Partial differential equations
3. Linear differential equations
4. Non-linear differential equations
5. Homogeneous and non-homogeneous differential equations

Variable Separable:

The first order ordinary equation of the form:

$$\frac{dy}{dx} = H(x.y) \tag{15}$$

Is called separable provided that $H(x.y)$can be written as the product of x and y. that is:

$$\frac{dy}{dx} = g(x)h(y) = \frac{g(x)}{f(y)}$$

Where $h(y) = \frac{1}{f(y)}$ and this implies that equation (7) becomes,

$$f(y)dy = g(x)dx \tag{16}$$

In principle it is easy to solve this special types of differential equation simply by taking integration both sides, then general solution is:

$$\int f(y)dy = \int g(x)dx + c$$

Where c is an arbitrary constant.

Results:

1. If f and $\frac{\partial f}{\partial y}$ are continuous near the initial point, there is a unique solution in a small interval around x_0.
2. First-order ODEs yield specific mathematical patterns when applied to physical systems.
3. A grid of small slopes that visually represent the "result" of the ODE without solving it algebraically.

Recommendations

1. If the equation is not separable but fits the form $y' + P(x)y = Q(x)$,use the integrating Factor method.
2. When using $\mu(x) = e^{\int P(x)dx}$, remember that $e^{\ln|f(x)|} = f(x)$. Simplifying this early makes the rest of the product rule integration much easier.
3. As $t \to \infty$, does your solution behave realistically? For example, in Newton's Law of Cooling, the temperature should approach the ambient temperature, not infinity.

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