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Supermodular HyperFunctions and Monotone HyperFunctions

Takaaki Fujita
Independent Researcher, Tokyo, Japan

Corresponding Author: **Takaaki Fujita**

Abstract
A hyperfunction associates each input with a set of admissible outputs, extending ordinary functions by permitting multivalued images rather than single values. A superhyperfunction uses iterated powersets for its domain and codomain, so it can encode hierarchical, multi-level output structure and hyperstructural multivalued behavior across a system. In this paper, we investigate supermodular hyperfunctions and monotone hyperfunctions, focusing on how these properties interact with such hierarchical, set-valued mappings.

Keywords: Hyperfunction, Superhyperfunction, Supermodular Function, Monotone Functions

1. Introduction
A function assigns each element of a domain exactly one value in a codomain, capturing deterministic input-output relationships between variables (Bylinski,1990) [12]. A hyperfunction maps each element to a set of possible outputs, generalizing classical functions by allowing multi-valued images within the domain (Fujita *et al*, 2025) [7]. An n-superhyperfunction maps subsets from iterated powersets to higher-level powersets, enabling hierarchical, multi-level outputs and hyperstructural multi-valued behavior across domains (Smarandache, 2022) [15]. These concepts have been actively studied in a variety of recent research works (Jdid *et al*.. 2025) [10].
Although HyperFunctions and SuperHyperFunctions are important concepts capable of representing hierarchical functional behavior, it cannot yet be said that they have been extensively studied. In this paper, we examine supermodular functions and monotone functions in the context of hyperfunctions and superhyperfunctions. These extended function frameworks may offer new approaches to applying supermodular and monotone function properties to hierarchical and layered structures.

2. Preliminaries
This section presents the key concepts and definitions required for the discussions in this paper. Unless otherwise stated, all sets and structures considered here are assumed to be finite and simple (undirected, no loops).

2.1 Hyperfunction and *n*-Superhyperfunction
Within the study of hyperstructures (Davvaz *et al*., 2018) [5] and n-superhyperstructures (Smarandache, 2024) [17] for functions, the notions of hyperfunction and n-superhyperfunction were formulated by Smarandache (Smarandache, 2022) [15]. Since then, hyperfunctions have attracted substantial attention and a variety of applications have been explored. For completeness, the essential definitions and related theorems are summarized below.

Definition 2.1 (Base Set). A base set X is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$X = \{x \mid x \text{ is an element within a specified domain}\}.$$

Definition 2.2 (Powerset) (Fujita, 2025b) [8]**.** The powerset of a set X, denoted $P(X)$, is the collection of all possible subsets of X, including both the empty set and X itself. Formally, it is expressed as:

$$P(X) = \{A \mid A \subseteq X\}.$$

Definition 2.3 (*n*-th Powerset) (Smarandache *et al*., 2022) The *n*-th powerset of a set *X*, denoted $P_n(X)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(X) = P(X);\ P_{n+1}(X) = P(P_n(X)), \text{ for } n \geq 1.$$

Similarly, the *n*-th non-empty powerset, denoted $P^*_n(X)$, is defined recursively as:

$$P^*_1(X) = P(X) \setminus \{\emptyset\};\ P^*_{n+1}(X) = P(P^*_n(X)) \setminus \{\emptyset\}.$$

Definition 2.4 (Hyperoperation) (Spartalis, 1996) [18]**.** A hyperoperation is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set *X*, a hyperoperation ∘ is defined as:

$$\circ: X \times X \to P(X).$$

Definition 2.5 (Hyperfunction) (Fujita *et al*, 2025) [7]**.** A Hyperfunction is a function where the domain remains a classical set *X*, but the codomain is extended to the powerset of *X*, denoted $P(X)$. Formally, a Hyperfunction *f* is defined as:

$f: X \to P(X)$. For any $x \in X$, $f(x) \subseteq X$ is a subset of *X*.

Example 2.6 Let $X = \{a, b, c\}$. Define $f(a) = \{a, b\}$, $f(b) = \{b\}$, $f(c) = \emptyset$. Here, input a "branches" to two admissible outputs, input *b* returns a single output, and input *c* yields no admissible output (e.g., a rule-based recommender that lists all acceptable next states for a given state).

Definition 2.7 (SuperHyperOperations). Let *X* be a non-empty set, and let $P_k(X)$ be the *k*-th powerset of *X*. Define:

$$P^0(H) = H,\ P^{k+1}(H) = P(P^k(H)) \text{ for } k \geq 0.$$

A SuperHyperOperation of order (m, n) is an *m*-ary operation:

$$\circ^{(m, n)}: H^m \to P^*_n(X).$$

If the codomain excludes the empty set, it is classical-type; if it includes it, it is Neutrosophic-type.

Definition 2.8 (*n*-Superhyperfunction) (Smarandache, 2022) [15]**.** An *n*-Superhyperfunction generalizes the concept of a Hyperfunction by using the *n*-th powerset $P_n(S)$ as the codomain. Formally, for $n \geq 2$, an *n*-Superhyperfunction f is defined as:

$$f: P_r(S) \to P_n(S),$$

Where $0 \leq r \leq n$, and $P_n(S)$ is the *n*-th powerset of *S*. This definition allows *f* to map subsets of *S* (from $P_r(S)$) to elements in the *n*-th powerset $P_n(S)$.

Example 2.9 (n = 2, r = 1): Let $S = \{1, 2\}$. Then $P_1(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and $P_2(S) = P(P_1(S))$. Define:

- $H(\emptyset) = \{\emptyset\}$
- $H(\{1\}) = \{\{1\}, \{1,2\}\}$
- $H(\{2\}) = \{\{2\}\}$
- $H(\{1,2\}) = \{\{1\}, \{2\}, \{1,2\}\}$.

Here, each input subset (from $P_1(S)$) is mapped to a set of subsets of $P_1(S)$ (an element of $P_2(S)$), modeling hierarchical, multi-level outcomes (e.g., mapping a chosen team to all admissible "teams-of-subtasks" configurations).

2.2 Supermodular Set Function

A supermodular set function exhibits increasing marginal gains; for any sets *A* and *B*, union plus intersection value dominates sum (Contreras *et al*., 2014 and Liberty *et al*., 2017) [4, 11].

Definition 2.10 (Contreras *et al*., 2014) [4]**.** Let *N* be a finite ground set and $f: 2^N \to \mathbb{R}$. The function f is supermodular if, for all subsets $A, B \subseteq N$,

$$f(A) + f(B) \leq f(A \cup B) + f(A \cap B).$$

Equivalently (marginal form), for all $A \subseteq B \subseteq N$ and any element $i \notin B$,

$$f(A \cup \{i\}) - f(A) \leq f(B \cup \{i\}) - f(B);$$

That is, marginal gains are increasing.

2.3 Monotone Set Functions

A monotone set function is nondecreasing: whenever A is contained in B, the function value never decreases when adding elements (Benvenuti *et al*, 2002) [3].

Definition 2.11 (Benvenuti *et al*, 2002) [3]**.** Let *N* be a finite ground set and $f: 2^N \to \mathbb{R}$. The function f is monotone (nondecreasing) if, for all $A \subseteq B \subseteq N$, $f(A) \leq f(B)$.
(Analogously, *f* is monotone nonincreasing if $f(A) \geq f(B)$ whenever $A \subseteq B$).

3. Main Results

This section presents the main contributions of the paper. Specifically, we examine the structures of Supermodular HyperFunctions, Supermodular SuperHyperFunctions, Monotone HyperFunctions, and Monotone SuperHyperFunctions.

3.1 Supermodular HyperFunction

A supermodular hyperfunction maps each collection of inputs to a set of outputs and "rewards pooling": considering two input collections together yields at least as comprehensive an output as processing them separately and then combining results.

Definition 3.1 Let *X* and *Y* be finite sets. A set hyperfunction is a map $H: 2^X \to 2^Y$ (multi-valued on subsets). *H* is supermodular if for all $A, B \subseteq X$,

$$H(A) \cup H(B) \subseteq H(A \cup B) \cup H(A \cap B).$$

Equivalently, for every monotone valuation $v: 2^Y \to \mathbb{R}$, the composite $v \circ H$ is a supermodular set function on 2^X.

Theorem 3.2 (Supermodular HyperFunctions generalize Supermodular Functions)
Let $H: 2^X \to 2^Y$ be a supermodular hyperfunction; that is, for all $A,B \subseteq X$, $H(A) \cup H(B) \subseteq H(A \cup B) \cup H(A \cap B)$.

Let $v: 2^Y \to \mathbb{R}$ be a monotone valuation (i.e., for all $U,V \subseteq Y$,

$v(U) + v(V) = v(U \cup V) + v(U \cap V)$, and $U \subseteq V$ implies

$v(U) \le v(V)$).

Define $f = v \circ H: 2^\wedge X \to \mathbb{R}$. Then f is a supermodular set function:

For all $A,B \subseteq X$, $f(A) + f(B) \le f(A \cup B) + f(A \cap B)$.

Proof. Fix $A,B \subseteq X$. By monotonicity of v,

$$v(H(A)) + v(H(B)) \le v(H(A) \cup H(B)) + v(H(A) \cap H(B)).$$

By supermodularity of H, $H(A) \cup H(B) \subseteq H(A \cup B) \cup H(A \cap B)$.

Applying monotonicity again and then the valuation identity,

$$v(H(A) \cup H(B)) + v(H(A) \cap H(B)) \le v(H(A \cup B) \cup H(A \cap B)) + v(H(A \cup B) \cap H(A \cap B)) = v(H(A \cup B)) + v(H(A \cap B)).$$

Combining the inequalities gives,

$$f(A) + f(B) \le f(A \cup B) + f(A \cap B),$$

So f is supermodular. ∎

3.2 Supermodular SuperHyperFunction
A supermodular superhyperfunction operates on hierarchical collections of sets and follows the same pooling principle across levels, so joint consideration of inputs never loses output compared with separate handling.

Definition 3.3 Let S be a finite set and $0 \le r \le n$. An n-superhyperfunction is a map $H: P_r(S) \to P_n(S)$, where $P_k(S)$ is the k-th iterated powerset. H is supermodular if for all $A,B \in P_r(S)$,

$$H(A) \cup H(B) \subseteq H(A \cup B) \cup H(A \cap B).$$

Equivalently, for any monotone valuation $v: P_n(S) \to \mathbb{R}$, the composite $v \circ H$ is a supermodular set function on $P_r(S)$.

Theorem 3.4 Fix a finite base set S and integers $0 \le r \le n$. An n-superhyperfunction is a map $H: P_r(S) \to P_n(S)$, where $P_k(S)$ is the k-fold iterated powerset. Say H is supermodular if for all $A,B \in P_r(S)$,

$$H(A) \cup H(B) \subseteq H(A \cup B) \cup H(A \cap B).$$

Then:
(a) Every supermodular hyperfunction is a special case of a supermodular superhyperfunction (take $r = n = 1$, so domain and codomain are ordinary powersets).
(b) Conversely, any supermodular superhyperfunction with $r = n = 1$ is exactly a supermodular hyperfunction.
Proof. When $r = n = 1$, we have $P_r(S) = 2^S$ and $P_n(S) = 2^S$, and the supermodularity condition above is identical to the hyperfunction condition. Thus the 1-level notion coincides with supermodular hyperfunctions, and the n-level notion (for arbitrary r,n) contains the 1-level case as a specialization. Hence supermodular superhyperfunctions strictly generalize supermodular hyperfunctions. ∎

3.3 Monotone HyperFunction
A monotone hyperfunction respects inclusion: whenever one input collection extends another, its produced output set also extends, never shrinking when the input grows.

Definition 3.5 Let $(X, \preceq)$ be a poset and let $f: X \to \mathcal{P}(S)$ be a hyperfunction (multi-valued map). The map f is monotone if for all $x, y \in X$ with $x \preceq y$, we have $f(x) \subseteq f(y)$; i.e., f is nondecreasing with respect to the domain order and set-inclusion in the codomain.

Theorem 3.6 (Monotone HyperFunctions generalize Monotone Functions)
Let $(X, \le_X)$ and $(Y, \le_Y)$ be posets. Every monotone function $g: X \to Y$ can be represented canonically as a monotone hyperfunction $H: X \to 2^Y$ by,

$$H(x) := \downarrow g(x) = \{y \in Y: y \le_Y g(x)\}.$$

Moreover, g is recovered from H by taking the maximum element of $H(x)$, so the embedding is faithful.
Proof. If $x \le_X y$ and g is monotone, then $g(x) \le_Y g(y)$. Hence $\downarrow g(x) \subseteq \downarrow g(y)$, so H is monotone as a hyperfunction. For each x, $H(x)$ is the principal ideal generated by $g(x)$, whose maximum is exactly $g(x)$; thus $x \mapsto max(H(x))$ reproduces g. Therefore the class of monotone functions sits inside the class of monotone hyperfunctions (as those whose images are principal ideals), proving that monotone hyperfunctions generalize monotone functions. ∎

3.4 Monotone SuperHyperFunction
A monotone superhyperfunction preserves inclusion throughout the hierarchy: richer or larger inputs at any level cannot lead to smaller outputs at that or higher levels.

Definition 3.7 Let S be a set, $0 \le r \le n$, and let $f: \mathcal{P}^r(S) \to \mathcal{P}^n(S)$ be an n-superhyperfunction. Using set-inclusion on both domain and codomain, f is monotone if for all $A, B \subseteq \mathcal{P}^{r-1}(S)$ with $A \subseteq B$, we have $f(A) \subseteq f(B)$; equivalently, f is isotone under inclusion.

Theorem 3.8 (Monotone SuperHyperFunctions generalize Monotone HyperFunctions).
Let *$(X, \le_X)$ be a poset and let $h: X \to 2^Y$ be a monotone hyperfunction. Define the union-lift:*

$$H^\vee: 2^X \to 2^Y \text{ by } H^\vee(A) := \bigcup\{x \in A\}\ h(x).$$

Then $H^\vee$ is a monotone 1-superhyperfunction (domain and codomain ordered by inclusion). Conversely, for any monotone 1-superhyperfunction $F: 2^X \to 2^Y$ that is union-preserving (i.e., $F(A) = \bigcup_{x \in A} F(\{x\})$ for all $A \subseteq X$), the restriction $f(x) := F(\{x\})$ is a monotone hyperfunction and F

= *(union-lift of f)*. Hence monotone superhyperfunctions strictly extend monotone hyperfunctions.

Proof. (Forward.) If $A \subseteq B$ then $\bigcup_{x \in A} h(x) \subseteq \bigcup_{x \in B} h(x)$, so $H^{\vee}$ is monotone under inclusion. Thus $H^{\vee}$ is a *1*-superhyperfunction. Also, for every x, $H^{\vee}(\{x\}) = h(x)$, so h is realized as the singleton restriction of $H^{\vee}$.

(Backward.) Suppose F is monotone and union-preserving. Define $f(x) := F(\{x\})$. For $x \leq_X$ y, the set inclusion $\{x\} \subseteq \{x,y\}$ and monotonicity of F give $F(\{x\}) \subseteq F(\{x,y\})$. Symmetrically, $F(\{y\}) \subseteq F(\{x,y\})$, hence f is isotone whenever $\leq X$ *refines inclusion over the singleton embedding (e.g., when* $\leq_X$ is the discrete order or when one works with the natural power-set domain). By union-preservation, for any A we have $F(A) = \bigcup_{x \in A} F(\{x\}) = \bigcup_{x \in A} f(x)$, so F is exactly the union-lift of f. Therefore every monotone hyperfunction embeds into a monotone superhyperfunction via union-lift, and union-preserving superhyperfunctions reduce to hyperfunctions by singleton restriction. ■

4. Conclusion

In this paper, we examined supermodular functions and monotone functions within the frameworks of hyperfunctions and superhyperfunctions.

For future work, we plan to investigate extensions using Fuzzy Functions (Demirci, 1999) [6], Intuitionistic Fuzzy Functions (Tak, 2020) [19], Neutrosophic Functions (Hatip, 2020) [9], Neutrosophic HyperFunctions (Al-Odhari, 2025) [2], 2-Refined Neutrosophic Functions (Musa, 2025), Neutrobalanced functions (Pandey, 2022), and Plithogenic Functions (Alhasan, 2023), in order to further develop hierarchical and uncertainty-aware functional models.

5. Declaration of conflicting interest

The authors declare that there is no conflict of interest in this work.

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7. Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

8. Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

9. Code Availability

No code or software was developed for this study.

10. Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

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