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Exact Traveling Wave Solutions for the Space-Time Fractional Modified Third-Order KdV Equation Via an Improved Analytical Technique

[1] Farah Madhi Munef Almadhi, [2] Hanan A Alkhidhr
[1, 2] Department of Mathematics, College of Sciences, Qassim University, Buraydah, Saudi Arabia

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Corresponding Author: **Farah Madhi Munef Almadhi**

Abstract
This paper presents a novel application of the improved modified extended tanh-function method to derive exact traveling wave solutions for the space-time fractional modified third- order Korteweg-de Vries (KdV) equation. The fractional derivatives are considered in the con- formable fractional derivative form. The proposed method systematically reduces the nonlinear fractional partial differential equation to an ordinary differential equation, which is then solved using a generalized ansatz. A specific kink-type solitary wave solution is obtained in closed form. The results demonstrate the robustness and efficiency of the method for handling com- plex nonlinear fractional differential equations, providing valuable new solutions that enhance our understanding of wave propagation in fractional media.

Keywords: Space-Time Fractional mKdV Equation, Improved Modified Extended Tanh-Function Method, Exact Solutions, Solitary Waves, Fractional Calculus

1. Introduction
Nonlinear fractional partial differential equations (NLPDEs) play a pivotal role in modeling complex phenomena across various scientific disciplines, including fluid dynamics, plasma physics, optical fibers, and biological systems. The fractional calculus framework, incorporating memory and hered- itary properties [10], provides a more accurate description of such processes than its integer-order counterpart. Among these models, the space-time fractional modified third-order KdV equation holds significant importance due to its ability to describe wave propagation in dispersive media with fractional effects. Determining exact analytical solutions for such equations is crucial, as these solu- tions offer profound insights into the underlying physical mechanisms and serve as benchmarks for numerical simulations.
In recent years, numerous analytical techniques have been developed to derive exact solutions for NLPDEs. Notable methods include the (G'/G)-expansion method [9, 12, 16], the exp-function method [14], the Kudryashov method [2-4], the sine-Gordon expansion method [8], Hirota's bilinear method [6, 11, 13], and the extended tanh-function method [1, 15]. Each method possesses unique strengths in handling specific types of nonlinearities and dispersion relations. The improved mod- ified extended tanh-function method [18], utilized in this study, is particularly effective due to its systematic procedure and ability to generate a wide spectrum of solutions, including solitary waves, periodic solutions, and singular solutions.
The primary objective of this paper is to apply the improved modified extended tanh-function method to obtain novel exact traveling wave solutions for the space-time fractional mKdV equation. The fractional derivative is considered in the conformable sense, which satisfies many properties of standard integer-order calculus. The obtained kink-type solution is both novel and physically significant. The reliability of the method is verified by checking the solution against the original equation. Furthermore, graphical interpretations are provided to visualize the dynamic behavior of the wave. This study demonstrates that the proposed method is a potent mathematical tool for investigating nonlinear fractional differential equations.

2. Conformable Fractional Derivative
The conformable fractional derivative, introduced by Khalil *et al.* [7], offers a natural and compu- tationally tractable generalization of the standard derivative. For a function f: $[0, \infty) \to \mathrm{R}$, the conformable derivative of order $\alpha \in (0, 1]$ is defined as [17]:

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0.$$

If the limit exists, f is said to be α-differentiable. This definition satisfies the following properties:

- $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)\ for\ a, b \in R$
- $T_\alpha(t^p) = pt^{p-\alpha} forp \in R$
- $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$
- $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$

If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$ This property is frequently used to simplify fractional differential equations via a suitable transformation.

3. The Improved Modified Extended Tanh-Function Method

Consider a general nonlinear fractional PDE of the form:

$$H(u, u_t, u_x, u_{xx}, \cdots) = 0 \tag{1}$$

Where H is a polynomial in $u(x,t)$ and its fractional partial derivatives. The fundamental steps of the improved modified extended tanh-function method are outlined below.

3.1 Traveling Wave Reduction

We introduce the complex wave transformation:

$$u(x,t) = u(\zeta), \quad \zeta = \omega x - \lambda t, \tag{2}$$

Where k and ω are nonzero constants representing the wave number and frequency, respectively. Applying this transformation to Eq. (2) reduces it to an ordinary differential equation (ODE):

$$H(u, u', u'', u''', \dots) = 0 \tag{3}$$

Where the prime denotes the derivative with respect to ζ.

3.2 Solution Ansatz

The method assumes a solution in the finite series form:

$$u(\zeta) = \sum_{i=0}^{n} a_i Y^i + \sum_{i=1}^{n} b_i Y^{-i}, \tag{4}$$

Where a_i, b_i are constants to be determined, and n is a positive integer found by balancing the highest-order derivative with the nonlinear term in Eq. (3). The function $Y(\zeta)$ satisfies the auxiliary Riccati equation:

$$Y'(\zeta) = \varepsilon\sqrt{c_0 + c_1Y + c_2Y^2 + c_3Y^3 + c_4Y^4}, \quad \varepsilon = \pm 1 \tag{5}$$

Where c_j ($j = 0, \dots, 4$) are real constants. By choosing specific values for c_j, various fundamental solutions for $u(\zeta)$ can be recovered (e.g., hyperbolic, trigonometric, Jacobi elliptic functions). For instance, setting $c_1 = c_3 = 0$, $c_2 < 0$, $c_4 > 0$, and $c_0 = c_2^2/(4c_4)$ yields the kink solution:

$$u(\zeta) = \varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\,\zeta\right) \tag{6}$$

4. Application to the Space-Time Fractional mKdV Equation

We consider the space-time fractional modified third-order KdV equation [5]:

$$D_t^\alpha u(x,t) + p\,u^2(x,t)\,D_x^\beta u(x,t) + q\,D_x^{3\beta} u(x,t) = 0, \quad 0 < \alpha, \approx \beta \le 1 \tag{7}$$

Where p and q are nonzero constants. Applying the wave transformation:

$$u(x,t) = u(\zeta), \quad \zeta = \omega \frac{x^\beta}{\beta} - \lambda \frac{t^\alpha}{\alpha} \tag{8}$$

and using the properties of the conformable derivative, Eq. (7) reduces to the integer-order ODE:

$$-\lambda u'(\zeta) + \omega p u^2(\zeta) u'(\zeta) + \omega^3 q u'''(\zeta) = 0 \tag{9}$$

Integrating once with respect to ζ and setting the integration constant to zero gives:

$$3\omega^3 q u''(\zeta) - 3\lambda u(\zeta) + \omega p u^3(\zeta) = 0 \tag{10}$$

By balancing the highest-order linear term with the nonlinear terms in Eq.(10) we obtain:

$$n - 2 + r = 3n \tag{11}$$

Since $r = 4$, we have $n = 1$. Substituting $n = 1$ into, we have:

$$u(\zeta) = a_0 + a_1 Y + b_1 Y^{-1} \tag{12}$$

Substituting Eq. (12) and the auxiliary equation (5) into Eq. (10) yields a system of algebraic equations. Equating the coefficients of different powers of Y to zero, we obtain the following system:
Solving this system, we obtain a family of solutions. A particular non-trivial solution correspond- ing to kink-type waves is found by choosing parameters such that $c_1 = c_3 = 0$, $c_2 < 0$, $c_4 > 0$, and, $c_0 = \frac{c_2^2}{4c_4}$ This yields:

$$a_0 = 0, \quad a_1 = a_1, \quad \varepsilon = 1$$
$$b_1 = -\frac{c_2^2 a_1}{2c_4}, \quad \omega = -\frac{3a_1\sqrt{-\frac{8p^3q}{27}}}{4c_4^{1/2}pq}, \quad \lambda = \frac{c_2 a_1^3\sqrt{-\frac{8p^3q}{27}}}{2c_4^{3/2}q} \tag{13}$$

Substituting these parameters into the ansatz (12) and using the kink solution (6) for $u(\zeta)$, we obtain the exact kink-type solution for the fractional mKdV equation:

$$u(\zeta) = a_1\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\zeta\right) - \frac{c_2^2 a_1}{2c_4}\left(\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\zeta\right)\right)^{-1} \tag{14}$$

Where $\zeta = \omega\frac{x^\beta}{\beta} - \lambda\frac{t^\alpha}{\alpha}$. The constants a_1, c_2, c_4 are free parameters subject to $c_2 < 0$ and $c_4 > 0$ to ensure the solution is real-valued.

5. Discussion

A potent mathematical model for explaining nonlinear wave propagation in complex dispersive me- dia, where both spatial and temporal dynamics display anomalous diffusion or memory effects, is the Space–Time Fractional Modified Third-Order Korteweg–de Vries (KdV) Equation. The fractional version adds fractional derivatives in space and time, allowing for a more accurate representation of nonlocal interactions and hereditary behaviors inherent in many physical systems, in contrast to the classical KdV equation, which mainly uses integer-order derivatives to capture shallow water wave phenomena. Applications of this equation can be found in solid-state physics, fluid dynamics, plasma physics, and optical fibers, where the evolution of compactons, shock structures, and solitary waves differs from classical behavior. The model offers a more comprehensive explanation of energy dispersion and dissipation by integrating fractional calculus, which enables more accurate control and comprehension of intricate nonlinear dynamics.

In order to investigate the behavior of soliton solution (14) and evaluate the physical relevance of the solutions obtained by choosing appropriate values for unknown parameters, a two of Matlab- created graphs are presented. Figs. 1 and 2 show a singularity at $x = 0$. The behavior is symmetric around the singularity, with a fast climb to positive infinity on one side and a drop to negative infinity on the other. This is characteristic of a pole-type solution, often known as a singular soliton. We noticed that by decreasing the fractional orde α, β, the amplitude of the solution is decreasing.

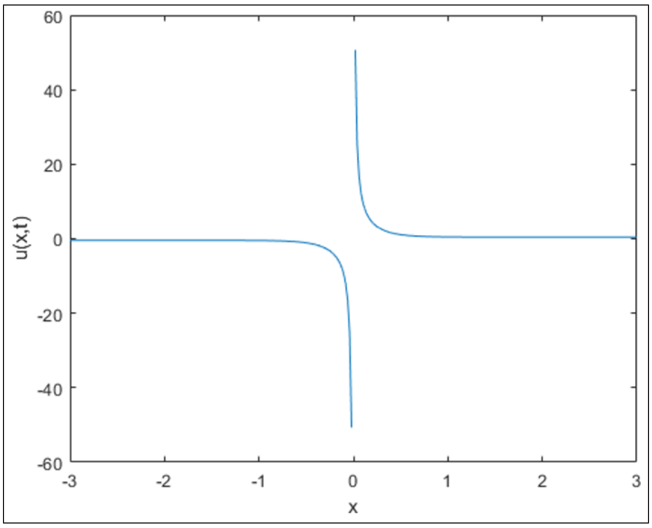

Fig 1: The 2D singular soliton corresponding to $u(x, t)$ with $\alpha = 1, \beta = 1$

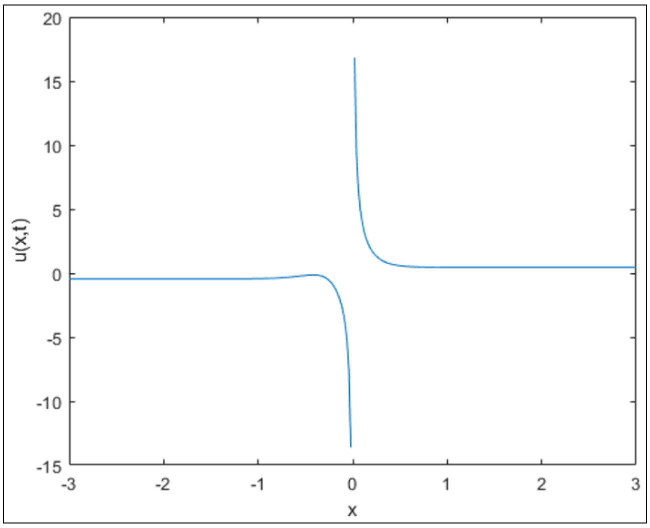

Fig 2: The 2D singular soliton corresponding to $u(x, t)$ with $\alpha = 0.8, \beta = 0.8$

6. Conclusion

The improved modified extended tanh-function method was successfully applied to the space-time fractional modified third-order KdV equation. The method efficiently reduces the fractional PDE to an ODE, and a new exact kink-type solitary wave solution was derived. The solution's structure was determined by selecting specific constants in the auxiliary equation, showcasing the method's adaptability and power. This work confirms the efficacy of the improved modified extended tanh-function method as a potent tool for extracting exact solutions from nonlinear fractional differential equations. The obtained solution provides a new analytical benchmark for understanding wave propagation in complex media with memory effects, such as viscoelastic materials or plasma waves. Future work could involve studying the dynamic properties of this solution and applying the method to other fractional integrable systems.

7. References

1. Adel M, Dumitru Baleanu, Umme Sadiya, Mohammad Asif Arefin, Hafiz Uddin M, Mahjoub A Elamin, *et al*. Inelastic soliton wave solutions with different geometrical structures to fractional order nonlinear evolution equations. Results in Physics. 2022; 38:105661.
2. Marwan Alquran. New interesting optical solutions to the quadratic-cubic schrodinger equation by using the kudryashov-expansion method and the updated rational sine-cosine functions. Optical and Quantum Electronics. 2022; 54(10):666.
3. Marwan Alquran. New symmetric bidirectional progressive surface-wave solutions to a general- ized fourth-order nonlinear partial differential equation involving second-order time-derivative. Journal of Ocean Engineering and Science, 2022.
4. Marwan Alquran, Feras Yousef, Farah Alquran, Tukur A Sulaiman, Abdullahi Yusuf. Dual- wave solutions for the quadratic–cubic conformable-caputo time-fractional klein-fock-gordon equation. Mathematics and Computers in Simulation. 2021; 185:62-76.
5. Mohammad Asif Arefin, Umme Sadiya, Mustafa Inc, Hafiz Uddin M. Adequate soliton solutions to the space-time fractional telegraph equation and modified third-order kdv equation through a reliable technique. Optical and Quantum Electronics. 2022; 54(5):309.

6. Ryogo Hirota, Junkichi Satsuma. Soliton solutions of a coupled korteweg-de vries equation. Physics Letters A. 1981; 85(8-9):407-408.
7. Roshdi Khalil, Mohammed Al Horani, Abdelrahman Yousef, Mohammad Sababheh. A new definition of fractional derivative. Journal of Computational and Applied Mathematics. 2014; 264:65-70.
8. Ayesha Khatun M, Mohammad Asif Arefin, Zohurul Islam M, Ali Akbar M, Hafiz Uddin M. New dynamical soliton propagation of fractional type couple modified equal-width and boussinesq equations. Alexandria Engineering Journal. 2022; 61(12):9949-9963.
9. Ayesha Khatun M, Mohammad Asif Arefin, Hafiz Uddin M, Mustafa I˙n¸c, Ali Akbar M. An analytical approach to the solution of fractional-coupled modified equal width and fractional- coupled burgers equations. Journal of Ocean Engineering and Science, 2022.
10. Urooj Sadia, Md. Ashik Arefin, Mustafa Inc, Md. Habib Uddin. Adequate soliton solutions to the space-time fractional telegraph equation and modified third-order KdV equation through a reliable technique. Preprints, 2021. Preprint.
11. Junkichi Satsuma, Ryogo Hirota. A coupled kdv equation is one case of the four-reduction of the kp hierarchy. Journal of the Physical Society of Japan. 1982; 51(10):3390-3397.
12. Imran Siddique, Mohammed MM Jaradat, Asim Zafar, Khush Bukht Mehdi, MS Osman. Exact traveling wave solutions for two prolific conformable m-fractional differential equations via three diverse approaches. Results in Physics. 2021; 28:104557.
13. Hon-Wah Tam, Wen-Xiu Ma, Xing-Biao Hu, Dao-Liu Wang. The hirota-satsuma coupled kdv equation and a coupled ito system revisited. Journal of the Physical Society of Japan. 2000; 69(1):45-52.
14. Hafiz Uddin M, Mohammad Asif Arefin, Ali Akbar M, Mustafa Inc. New explicit solutions to the fractional-order burgers' equation. Mathematical Problems in Engineering. 2021; 1:6698028.
15. Hafiz Uddin M, Md Ashrafuzzaman Khan, Ali Akbar M, Md Abdul Haque. Analytical wave solutions of the space time fractional modified regularized long wave equation involving the conformable fractional derivative. Karbala International Journal of Modern Science. 2019; 5(1):7.
16. Hafiz Uddin M, Ayesha Khatun M, Mohammad Asif Arefin, Ali Akbar M. Abundant new exact solutions to the fractional nonlinear evolution equation via riemann-liouville derivative. Alexandria Engineering Journal. 2021; 60(6):5183-5191.
17. Guanli Xiao, Jinrong Wang. Representation of solutions of linear conformable delay differ- ential equations. Applied Mathematics Letters. 2021; 117:107088.
18. Zhong Yang, Hon YC. An improved modified extended tanh-function method. Zeitschrift fu¨r Naturforschung A. 2006; 61(3-4):103-115.