Int. j. adv. multidisc. res. stud. 2025; 5(4):1738-1745

Received: 09-07-2025
Accepted: 19-08-2025

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

A Class of Multivalent Functions Based on a Linear Operator

[1] **Awoyale Olusegun,** [2] **Fatunsin Modupe Lolade,** [3] **Amao Atinuke Ayanfe**
[1] Department of Mathematics, Federal University of Education, Kontagora, Nigeria
[2] Department of Mathematics Programme, National Mathematical Centre Abuja, Nigeria
[3] Department of Mathematics, University of Ilorin, Nigeria

DOI: https://doi.org/10.62225/2583049X.2025.5.4.4842
Corresponding Author: **Awoyale Olusegun**

Abstract
In this work, we define a new class Mp (A, B, n, $\mu,\alpha,\theta,\varsigma$) of multivalent functions. We discuss coefficient estimates, distortion and growth properties, radii of close-to- convexity and starlikeness of the class. The integral transforms of the class are also obtained. We employ the principle of subordination and El-Ashwah linear operator to establish our results.

Keywords: Analytic Functions, Multivalent Functions, Subordination, Close-to-Convexity

1. Introduction

Let A denote the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$
Let *Mp* be the subclass of A consisting of analytic p-valent functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p},\ p \in N \tag{1}$$

The class *Mp* is called the class of multivalent (p-valents) functions.
A function $f(z)$ is called a *p*-valent function if it does not take a value more than *p*-times.
Following El-Ashwah [3], we write the following equalities for the functions;

f(z)∈Mp:

$$\zeta_p^t(\mu,\theta) f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{\mu + \theta(k+p)}{\mu}\right)^t b_{k+p} z^{k+p} \tag{2}$$

Where $\theta \geq 0$, $\mu > 0$, $t \in N_0 = N \cup 0$

We can write (2) in the form $\zeta_p^t(\mu,\theta) f(z) = \zeta_p^t \left(\psi_{\mu,\theta}^{p,t} * f\right)(z)$ where:

$$\psi_{\mu,\theta}^{p,t} f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{\mu + \theta(k+p)}{\mu}\right)^t b_{k+p} z^{k+p}$$

It is easily verified from (2) that:

$$\theta z \left(\zeta_p^t(\theta,\mu) f(z)\right)' = \mu \zeta^{t+1}(\theta,\mu) f(z) - (\theta p + \mu)\zeta^t f(z)$$

We note that:

$$\zeta^0(\theta,\mu)f(z) = f(z)$$

$$\zeta^1(1,1)f(z) = \frac{\left(z^{p+1f(z)}\right)'}{z^p} = (p+1)f(z) + zf'(z)$$

The class of multivalent functions has been widely studied. In recent years, several interesting classes (and or subclasses) of multivalent functions have been introduced and investigated (for examples, see [1, 2, 4, 5, 6, 7]).

Definition 1: For two functions s_1^* and s_2^* analytic in *U*, we say that the function s_1^* (z) is subordinate to s_2^* (z) in *U* and write s_1^* $(z) \propto s_2^*$ (z), if there exists a Schwarz function $w(z)$, analytic in *U* with $w(0) = 0$ and $|w(z)| < 1$ such that s_1^* $(z) = s_2^*$ $(z) = s_2^*(w(z))(z \in U)$.

Definition 2: Let *Mp* (A, B, n, $\mu, \alpha, \theta, \varsigma$). be a class that consists of function $f(z)$ which satisfy the condition.

$$1 + \frac{1}{n}\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right) \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha \tag{3}$$

Where $\prec$ denotes the subordination, n is a complex number, A and B are the arbitrary fixed number $-1 \le B < A \le 1, 0 \le \alpha < 1 \text{ and } z \in U$.

The condition (3) is equivalent to:

$$\left|\frac{\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right)}{n(A-B)(1-\alpha) - B\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right)}\right| < 1 \tag{4}$$

Remark:
When $\mu = \varsigma = 1$, n = b and α = t = 0 in (3), we obtain the class $S_b^*(\varphi)$.
Also, when $\mu = \varsigma$ = t = 1, n = b and α = 0 in (3), we have a subclass $C_b(\varphi)$.
The two classes were studied by Ravichandran *et al*. [8]. In this work, we discuss coefficient estimates, growth and distortion properties. The radii of close-to-starlike and starlikeness of the class M_p (A, B, n, μ, α, θ, ζ) are also considered. Furthermore, the integral transform of the class is obtained.

2. Coefficient Estimates

Theorem 1: A function defined by (1) is in the class M_p (A, B, n, μ, α, θ, ζ) if and only if;

$$\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]b_k}{|n|(A-B)(1-\alpha)} \le 1 \tag{5}$$

Proof: Let $f(z) \in M_p$ (A, B, n, t, μ, α, θ, ζ), then by the definition of subordination (3) can be written as:

$$1 + \frac{1}{n}\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right) = (1-\alpha)\frac{1+Aw(z)}{1+Bw(z)} + \alpha \tag{6}$$

Where $|w(z)| < 1, |w(0)| = 1$

From the equation (6), we have:

$$\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right) = \frac{n(A-B)(1-\alpha)w(z)}{1+Bw(z)} \tag{7}$$

Upon expanding and rearranging (7), we get:

$$\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right) = \left\{n(A-B)(1-\alpha) - B\left(\frac{\zeta^{t+1}(\mu,\theta)f(z)}{\zeta^t(\mu,\theta)f(z)} - 1\right)\right\}w(z) \tag{8}$$

We apply the differential operator $\zeta^{t+1}(\mu,\theta)\, f(z) defined\ by\ (2) in\ (8) to\ get$:

$$\frac{\sum_{k=1}^{\infty}\left(\frac{\mu+\theta(k+p)}{\mu}\right)^{t}\left(\frac{\mu+\theta(k+p)}{\mu}\right)}{\sum_{k=1}^{\infty}\left(\frac{\mu+\theta(k+p)}{\mu}\right)^{t}}-1=\left\{n(A-B)(1-\alpha)-B\left(\frac{\sum_{k=1}^{\infty}\left(\frac{\mu+\theta(k+p)}{\mu}\right)^{t}\left(\frac{\mu+\theta(k+p)}{\mu}\right)}{\sum_{k=1}^{\infty}\left(\frac{\mu+\theta(k+p)}{\mu}\right)^{t}}-1\right)\right\}w(z).$$

That is;

$$\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta(k+p)}{\mu}\right]b_{k+p}=$$

$$\left\{n(A-B)(1-\alpha)-\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta B(k+p)}{\mu}+n(A-B)(1-\alpha)\right]b_{k+p}\right\}w(z) \tag{9}$$

Since $|\omega(z)| < 1$, then from (9) it follows that:

$$|w(z)|=\left|\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta(k+p)}{\mu}\right]b_{k+p}}{n(A-B)(1-\alpha)-\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta B(k+p)}{\mu}+n(A-B)(1-\alpha)\right]b_{k+p}}\right|<1$$

Since $Re(z) \leq |z|$ for all z, it follows that:

$$Re\left\{\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta(k+p)}{\mu}\right]b_{k+p}}{n(A-B)(1-\alpha)-\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta B(k+p)}{\mu}+n(A-B)(1-\alpha)\right]b_{k+p}}\right\}<1$$

Letting $z \to 1^-$ through the real values, and upon clearing the denominator, we obtain;

$$\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k+p}}{|n|(A-B)(1-\alpha)}\leq 1$$

Conversely, let the inequality (5) holds. Since $|\omega(z)| < 1$, from (4) we have:

$$\left|\frac{\frac{\zeta^{\delta+1}(\mu,\theta)f(z)}{\zeta^{\delta}(\mu,\theta)f(z)}-1}{n(A-B)(1-\alpha)-B\left(\frac{\zeta^{\delta+1}(\mu,\theta)f(z)}{\zeta^{\delta}(\mu,\theta)f(z)}-1\right)}\right|$$

$$=\left|\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta(k+p)}{\mu}\right]b_{k+p}z^{k+p}}{n(A-B)(1-\alpha)-\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta B(k+p)}{\mu}+n(A-B)(1-\alpha)\right]b_{k+p}z^{k+p}}\right|<1$$

$$\leq\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta(k+p)}{\mu}\right]b_{k+p}}{|n|(A-B)(1-\alpha)-\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k+p}}$$

The last expression is bounded by 1 if:

$$\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta(k+p)}{\mu}\right]b_{k+p}\leq|n|(A-B)(1-\alpha)-\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k+p}$$

Thus

$$\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k+p}}{|n|(A-B)(1-\alpha)}\leq 1$$

3. Growth and Distortion Theorems

We now prove the growth theorem for the functions in the class *Mp*(A,B,b,n,t,μ,α, θ,ζ)

Theorem 2: Let the f(z) defined by (1) belong to the class *Mp* (A,B,b,n,t,μ,α, θ,ζ) then $0 < |= r < 1z|$:

$$r^{p}-r^{p+1}\left\{\frac{|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\}\leq f(z)\leq$$
$$r^{p}+r^{p+1}\left\{\frac{|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\}$$

$$f(z)=z^{p}+\left\{\frac{|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k}}\right\}z^{p+1},(z=r).$$

Proof: Let $f(z)\in$ *Mp*(A,B,b,n,t,μ,α, θ,ζ), then from Theorem 1, it is noted that:

$$\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k+p}\leq|n|(A-B)(1-\alpha)$$

Since:

$$\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]b_{k+p}$$

is an increasing function of k, it implies that

$$\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]\sum_{k=1}^{\infty}b_{k+p}\leq|n|(A-B)(1-\alpha)$$

Hence:

$$\sum_{k=1}^{\infty}b_{k+p}\leq\frac{|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\lambda(k+p)}{\mu}\right]^{t}\left[\frac{2\lambda B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]} \tag{10}$$

Also from equations (1) and (10) and for $|z|=r$ it follows that:

$$|f(z)|\leq r^{p}+\sum_{k=1}^{\infty}r^{k}b_{k+p}$$
$$|f(z)|\leq r^{p}+r^{p+1}\sum_{k=1}^{\infty}b_{k+p}$$
$$\leq r^{p}+r^{p+1}\left\{\frac{|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^{t}\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\}$$

Similarly,

$$|f(z)|\geq r^{p}-r^{p+1}\sum_{k=1}^{\infty}b_{k+p}$$

$$\geq r^p - r^{p+1}\left\{\frac{|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\}$$

This completes the proof of Theorem 2

Taking A = p = μ = θ = 1 and B = -1 in Theorem 2,we get

Corollary 1: $If\ f(z) \in M(n,\alpha,t)$, then

$$r - \frac{|n|(1-\alpha)}{(2+|n|(1-\alpha))\,3^{\delta}}r^2 \leq |f(z)| \leq r + \frac{|n|(1-\alpha)}{(2+|n|(1-\alpha))\,3^{\delta}}r^2.$$

Theorem 3

Let $f(z)$ defined by (1) belong to the class *Mp* (A,B,b,n,t,μ,α, θ,ζ), then for $0 < |z| = r < 1$,

$$pr^{p-1} - r^p\left\{\frac{(p+1)|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\} \leq |f'(z)| \leq$$
$$pr^{p-1} + r^p\left\{\frac{(p+1)|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\} \qquad (11)$$

Proof: From (10),

$$\sum_{k=1}^{\infty} kb_{k+p} \leq \left\{\frac{(p+1)|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]|b_{k+p}|}\right\} \qquad (12)$$

Combining (1) and (12) and for $|z| = r$ it is obtained that

$$|f'(z)| \leq pr^{p-1} + \sum_{k=1}^{\infty}(k+p)b_{k+p}r^{k-1}|$$
$$\leq pr^{p-1} + r^p\sum_{k=1}^{\infty}(k+p)b_{k+p}$$
$$\leq pr^{p-1} + r^p\left\{\frac{(p+1)|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{l}+|n|(A-B)(1-\alpha)\right]}\right\}$$

Similarly,

$$|f'(z)| \geq pr^{p-1} - \sum_{k=1}^{\infty}(k+p)b_{k+p}r^{k-1}|$$
$$\geq pr^{p-1} - r^p\sum_{k=1}^{\infty}(k+p)b_{k+p}$$
$$\geq pr^{p-1} - r^p\left\{\frac{(p+1)|n|(A-B)(1-\alpha)}{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu}+|n|(A-B)(1-\alpha)\right]}\right\}$$

This ends the proof of Theorem 3

Taking A = p = $\mu = \theta$ = 1 and B = -1 in Theorem 3,we get

Corollary 2: If $f(z) \in M(n,\alpha,t)$, then,

$$r - \frac{|n|(1-\alpha)}{(2+|n|(1-\alpha))\,3^{t}}r^2 \leq |f(z)| \leq r + \frac{|n|(1-\alpha)}{(2+|n|(1-\alpha))\,3^{t}}r^2$$

Which completes the proof.

4. Radius of Close-to-convex for the class *Mp* (A,B,b,n,t,μ, α, θ, ζ)

Theorem 4. Let $f\ (z)$ defined by (1) belong to the class *Mp* (A,B,b,n,t,μ, α, θ, ζ), then $f\ (z)$ is *p*-valent close-to-convex of order $\beta\ \ (0 \le \beta < 1)$ in the disk $|z| = r_1$ where:

$$r_1 = \inf_k \left[\frac{(p-\beta)\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]}{(k+p)|n|(A-B)(1-\alpha)} \right]^{\frac{1}{k}} \tag{13}$$

Proof: It is suffice to show that:

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p - \beta$$

Suppose $f\ (z) \in$ $M_p(A, B, n, t, \mu, \alpha, \beta, \theta, \zeta)$, using (1) it shows that:

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} + \sum_{k=p+1}^{\infty}(k+p)b_k z^{k+p-1} - pz^{p-1}}{z^{p-1}}\right| \tag{14}$$

Implies

$$\left|\sum_{k=p+1}^{\infty}(k+p)b_k z^k\right| \le \sum_{k=p+1}^{\infty}(k+p)|b_k||z^k|$$

then

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p - \beta$$

if $$\frac{\sum_{k=p+1}^{\infty}(k+p)\,|b_k|\,|z|^{k-p}}{p-\beta} < 1 \tag{15}$$

By Theorem 1

$$\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\lambda(k+p)}{\mu}\right]^t \left[\frac{2\lambda B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]|b_k|}{|n|(A-B)(1-\alpha)} \le 1 \tag{16}$$

Thus (16) is true if

$$\frac{\sum_{k=p+1}^{\infty}(k+p)\,|b_k|\,|z|^{k-p}}{p-\beta} < \frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]|b_k|}{|n|(A-B)(1-\alpha)} \tag{17}$$

Solving for z in (17), we have:

$$|z| < \left[\frac{(p-\beta)\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]}{(k+p)|n|(A-B)(1-\alpha)} \right]^{\frac{\mu}{k}} \tag{18}$$

Which completes the proof.

5. Radius of Starlikeness of the class $M_p(A, B, n, t, \mu, \alpha, \beta, \theta, \zeta)$,

Theorem 5. Let the function defined by (1) belong to the class $M_p(A, B, n, t, \mu, \alpha, \beta, \theta, \zeta)$, then $f\ (z)$ is p-valent starlike of order $\beta\ \ (0 \le \beta < 1)$ in the disk $|z| < r_2$,

Where

$$r_2 = \inf_k \left[\frac{(p-\beta)\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]}{(k+p-\beta)|n|(A-B)(1-\alpha)} \right]^{\frac{1}{k}}$$

Proof: We must show that:

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \beta, for|z| < r_2$$

Indeed from (1)

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=1}^{\infty} k\,|b_{k+p}|\,|z|^k}{1 - \sum_{k=p+1}^{\infty} |b_{k+p}|\,|z|^k}$$

Thus,

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \beta$$

if $$\frac{\sum_{k=1}^{\infty}(k + p - \beta)\,|b_{k+p}|\,|z|^k}{p - \beta} < 1 \qquad (19)$$

But Theorem 1 assures that:

$$\frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]|b_{k+p}|}{|n|(A-B)(1-\alpha)} \leq 1 \qquad (20)$$

In view of (20) it follows that (19) will be true if

$$\frac{\sum_{k=p+1}^{\infty}(k + p - \beta)\,|b_{k+p}|\,|z|^k}{p - \beta} < \frac{\sum_{k=1}^{\infty}\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]|b_{k+p}|}{|n|(A-B)(1-\alpha)} \qquad (21)$$

Solving for $|z|$ in (21), we have

$$|z| < \left[\frac{(p-\beta)\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t\left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right]}{(k+p-\beta)|n|(A-B)(1-\alpha)}\right]^{\frac{\mu}{k}}$$

This complete the proof of Theorem 4

6. Integral Operator

In this section, the integral transforms of function in the class $M_p(A, B, n, t, \mu, \alpha, \beta, \theta, \zeta)$, are discussed.

Theorem 6

Let the function $f(z)$ given by (1) be in the class $M_p(A, B, n, t, \mu, \alpha, \beta, \theta, \zeta)$. Then the integral transforms:

$$F(z) = \frac{c+p}{z^p}\int_0^z t^{c-1} f(t)dt, c > -p$$

are in the class $F_p{}^*(\gamma), 0 \leq \gamma < p$ where

$$\gamma = \gamma(A, B, p, n, \alpha, \beta) = \frac{(c+1)\frac{\theta B}{\mu}(p+2) + |n|(A-B)(1-\alpha) - \frac{\theta B}{\mu}(1-\alpha)(c+p)}{(c+1)\frac{\theta B}{\mu}(p+1) + |n|(A-B)(1-\alpha)}$$

The result is sharp for the function

$$f(z) = z^p + \frac{n(A-B)(1-\alpha)}{[(k-p)(1-B) + n(A-B)(1-\alpha)]\left(\frac{p+1}{p}\right)^{\delta}} z^{p+1} \qquad (22)$$

Proof:
Let

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in M_p(A, B, n, \mu, \delta, \alpha, \theta, \zeta).$$

Then

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt, c > -p$$

$$= z^p + \sum_{k=p+1}^{\infty} \left(\frac{c+p}{c+k}\right) b_{k+p} z^k$$

In view of Theorem 1,we shall find the largest γ for which

$$\frac{\sum_{k=p+1}^{\infty} \left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\gamma)\right] b_{\mathbf{k+p}}}{|n|(A-B)(1-\gamma)} \left(\frac{c+p}{c+k}\right) \leq 1$$

It is sufficient to find the range values of γ for each k ∈ N

$$\frac{\left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\gamma)\right] |b_k|}{|n|(A-B)(1-\gamma)} \left(\frac{c+p}{c+k}\right) \leq \frac{\sum_{k=p+1}^{\infty} \left[\frac{\mu+\theta(k+p)}{\mu}\right]^t \left[\frac{2\theta B(k+p)}{\mu} + |n|(A-B)(1-\alpha)\right] |b_{k+p}|}{|n|(A-B)(1-\alpha)} \tag{23}$$

Solving for γ in (23), we have

$$\gamma \leq \frac{(c+k)\frac{\theta B}{\mu}(k+p) + |n|(A-B)(1-\alpha)(k-p) - \frac{\theta B}{\mu}(k+p)(1-\alpha)(c+p)}{(c+k)\frac{\theta B}{t}(k+p) + |n|(A-B)(1-\alpha)(k-p)} \tag{24}$$

Since the right hand side of (24) is an increasing function of k, putting k = 1
in (24), we get

$$\gamma \leq \frac{(c+1)\frac{2\theta B}{\mu}(p+1) + |n|(A-B)(1-\alpha) - \frac{2\theta B}{\mu}(p+1)(1-\alpha)(c+p)}{(c+1)\frac{2\theta B}{\mu}(p+2) + |n|(A-B)(1-\alpha)}$$

This complete the proof of the Theorem.

7. References

1. Aouf MK, El-Deeb SM. On p-valent strongly starlike and strongly convex functions associated with generalised linear operator. Montes Taurus Journal of pure and Applied Mathematics. 2021; 3(3):104-111.
2. Awoyale O, Opoola TO, Makinde DO. A New class of p-valent functions defined by an extended Salagean differential operator. Open Journal of Mathematical Analysis. 2024; 8:208-215.
3. El-Ashwah RM. A note on certain meromorphic p-valent functions, Appl. Math. Letters. 2009; 22:1756-1759.
4. Almajar O, Alshammari K, Amourah AM, Dairus M. A specific category of meromorphic functions with positive coefficients. European Journal of Pure and Applied Mathematics. 2025; 18(2), art:6107.
5. Gambo Y, Babalola KO, Mustapha AU, Emmanuel DE. Some properties of analytical and p-valent functions involving certain convolution operators. International Journal of Mathematics Trends and Technology. 2021; 67(10):34-40.
6. Sambo B, Opoola TO. Geometric properties of new subclasses of analytic functions defined by Opoola differential operator. Open Journal of Mathematical Analysis. 2024; 8(2):55-64.
7. Techranchi A, Kulkarni SR, Murugusundaramoorthy G. On certain of p-valent functions defined by multplier transformation and differential operator. Journal of Indonesian of Mathematical Society. 2007; 13(1):27-38.
8. Ravichandran V, Polatoglu Y, Bolcal M, Arzu S. Certain subclasses of starlike and convex functions of complex order. Hacettepe Journal of Mathematics and Statistics. 2005; 34:9-15.