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The solution of the Problem Arising in the Displacement Process in Double Phase Due to Differential Wettability

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Abstract

This paper presents a numerical solution for the one-dimensional flow of two immiscible fluids, 'oil' and 'water', in a homogeneous porous medium. Water is injected uniformly at a constant velocity into the unsaturated section of this medium, causing it to displace the fluids over a short distance. Our objective is to obtain a numerical solution for phase saturation distribution. The mathematical analysis leads to a governing equation, a second-order linear partial differential equation, which is solved using an explicit scheme under suitable boundary conditions. The porosity and permeability of the medium are treated as constant. The numerical solutions to this governing equation are presented in tables and graphs generated using MATLAB.

Keywords: Displacement, Porous Media, Phase Saturation, Finite Difference Method, Differential Wettability

1. Introduction

It's widely recognised that when two or more immiscible fluids or phases (like oil, water, and gas) flow simultaneously through a porous medium, there's a non-zero interfacial tension between them. Consequently, a distinct fluid-fluid interface separates the fluids within each pore. At every point on this interface, a capillary pressure difference exists [1].

The first mathematical formulation of the flow of two immiscible liquids in a homogeneous porous medium was presented by Buckley-Leverett [2]. However, their solution was obtained under simplifying assumptions. R.S. Patel [3]. and S.S. Patel [4] made a formal attempt to discuss the two-phase flow problem in the presence of capillary pressure. Verma [5, 6] obtained an approximate solution to the two-phase flow problem in a heterogeneous porous medium under specific conditions, considering the effects of capillary pressure and variations in phase density separately. Many researchers, including M.R.Tailor [9], P.V.Tandel [10] and M.S.Prajapati [11] have discussed this phenomenon from different perspectives.

In this paper, we have analysed the one-dimensional flow of two immiscible fluids, 'oil' and 'water,' in a homogeneous porous medium. We have considered the analytical expressions for the relationship between relative permeability and phase saturation, as suggested by Jones Jones [7] and for capillary pressure, phase saturation by Oroveanu [8]. Our primary objective is to obtain numerical and graphical solutions for the phase saturation distribution. The numerical solutions to the governing equation are presented in tables and graphs generated using MATLAB.

2. Statement of the problem:

In the investigation problem considered here, when water is uniformly injected with a constant velocity into an unsaturated dipping of homogeneous porous medium, it is assumed that the entire oil on the initial boundary $(\alpha = 0)$ is displaced through a small distance due to the impact of injecting water.

The primary objective of this investigation is to obtain a numerical solution for phase saturation under the assumption that the porosity and permeability of the medium remain constant. The mathematical solution to the second-order linear partial differential equation governing the flow of two immiscible fluids is being obtained under specific conditions.

3. Fundamental equations:

The seepage velocity of the water and oil may be written as by Darcy's law,

$$V_w = -\frac{K_w}{\delta_w} K\left[\left(\frac{\partial P_w}{\partial X} + \rho_w \cdot g \cdot \sin\alpha\right)\right] \tag{3.1}$$

$$V_o = -\frac{K_o}{\delta_o} K \left[\left(\frac{\partial P_o}{\partial x} + \rho_o \cdot g \cdot \sin\alpha \right) \right] \tag{3.2}$$

Where

$K =$ permeability of the homogeneous medium, K_w & $K_o =$ relative permeability of water and oil, respectively which are the functions of S_w and S_o

S_w & $S_o =$ saturation of water and oil, respectively

P_w & P_o = pressure of water and oil respectively

δ_w and δ_o are the constant kinematics viscosities

$\alpha =$ the inclination of the bed

$g =$ gravitational constant

ρ_w and ρ_o are constant densities of water and oil respectively.

The equations of continuity for the two phases (assuming constant phase densities) are,

$$P\left(\frac{\partial S_w}{\partial t}\right) + \left(\frac{\partial V_w}{\partial x}\right) = 0 \tag{3.3}$$

$$P\left(\frac{\partial S_o}{\partial t}\right) + \left(\frac{\partial V_o}{\partial x}\right) = 0 \tag{3.4}$$

Where P is the porosity of the medium.

From the definition of phase saturation, it is evident that:

$$S_w + S_o = 1 \tag{3.5}$$

The capillary pressure P_c, in a double phase fluid flow is defined to be the pressure discontinuity between the flowing phases across their common interface. As per Scheidegger [12] it can be expressed as,

$$P_c = \beta(S_w^{-2} - C); \beta \text{ and } C \text{ are constants.} \tag{3.6}$$

$$P_c = P_o - P_w \tag{3.7}$$

Relation between relative permeability and phase saturation:

For the sake of mathematical clarity, we assume the standard relationships between capillary pressure, phase saturation, and relative permeability, as outlined in [12]

$$K_w = S_w^3 \tag{3.8}$$

$$K_o = 1 - \alpha_o S_w = S_o; \quad \alpha_o = 1.11 \tag{3.9}$$

The general flow equations

The equation of motion for saturation can be obtained by substituting the values of V_w and V_o form equations (3.1) and (3.2) into equations (3.3) and (3.4) respectively, we get

$$P\left(\frac{\partial S_w}{\partial t}\right) = \frac{\partial}{\partial x}\left[\frac{K_w}{\delta_w} K \left(\frac{\partial P_w}{\partial x} + \rho_w \cdot g \cdot \sin\alpha\right)\right] \tag{3.10}$$

$$P\left(\frac{\partial S_o}{\partial t}\right) = \frac{\partial}{\partial x}\left[\frac{K_o}{\delta_o} K \left(\frac{\partial P_o}{\partial x} + \rho_o \cdot g \cdot \sin\alpha\right)\right] \tag{3.11}$$

When pressure discontinuity and gravity terms are considered in an inclined porous medium, the general flow equations for the phases in the homogeneous medium are discussed here.

Instability with capillary pressure

Substituting the value of $\frac{\partial P_w}{\partial x}$ from (3.7) into (3.10), we obtain

$$P\left(\frac{\partial S_w}{\partial t}\right) = \frac{\partial}{\partial x}\left[\frac{K_w}{\delta_w} K \left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} + \rho_w \cdot g \cdot \sin\alpha\right)\right] \tag{3.12}$$

From the combination of equations (3.11) and (3.12) and using (3.5), we get;

$$\frac{\partial}{\partial x}\left[\left(\frac{K_w}{\delta_w}K+\frac{K_o}{\delta_o}K\right)\frac{\partial P_o}{\partial x}-\frac{K_w}{\delta_w}K\left(\frac{\partial P_c}{\partial x}\right)+\left(\frac{K_w}{\delta_w}K\rho_w+\frac{K_o}{\delta_o}K\rho_o\right)g\cdot\sin\alpha\right]=0 \tag{3.13}$$

Integrating (3.13) with respect to X, we obtain;

$$\left[\left(\frac{K_w}{\delta_w}K+\frac{K_o}{\delta_o}K\right)\frac{\partial P_o}{\partial x}-\frac{K_w}{\delta_w}K\left(\frac{\partial P_c}{\partial x}\right)+\left(\frac{K_w}{\delta_w}K\rho_w+\frac{K_o}{\delta_o}K\rho_o\right)g\cdot\sin\alpha=-V\right] \tag{3.14}$$

Where V is the integrating constant which can be evaluated later on.
Simplification of (3.14)

$$\frac{\partial P_o}{\partial x}=\frac{1}{\left(1+\frac{K_o\delta_w}{K_w\delta_o}\right)}\frac{\partial P_c}{\partial x}-\frac{V}{\left(\frac{K_w}{\delta_w}K+\frac{K_o}{\delta_o}K\right)}-\frac{\rho_w+\frac{K_o\delta_w}{K_w\delta_o}\rho_o}{\left(1+\frac{K_o\delta_w}{K_w\delta_o}\right)}\cdot g\cdot\sin\alpha \tag{3.15}$$

Using (3.15) and (3.12), we have

$$P\left(\frac{\partial S_w}{\partial t}\right)+\frac{\partial}{\partial x}\left[\frac{\frac{K_o}{\delta_o}}{\left(1+\frac{K_o\delta_w}{K_w\delta_o}\right)}\left(\frac{\partial P_c}{\partial x}-(\rho_w-\rho_o)\cdot g\cdot\sin\alpha\right)+\frac{V}{\left(1+\frac{K_o\delta_w}{K_w\delta_o}\right)}\right]=0 \tag{3.16}$$

The value of the pressure of oil (P_o) can be written as in Oroveanu [8] in the form:

$$P_o=\frac{P_o+P_w}{2}+\frac{P_o-P_w}{2}=\bar{P}+\frac{1}{2}P_c \tag{3.17}$$

Where $\bar{P}$ is the mean pressure which is constant, equation (3.17) written as:

$$\frac{\partial P_o}{\partial x}=\frac{1}{2}\frac{\partial P_c}{\partial x} \tag{3.18}$$

Using (3.18) and (3.14), we get

$$V=\frac{1}{2}\left(\frac{K_w}{\delta_w}K-\frac{K_o}{\delta_o}K\right)\frac{\partial P_c}{\partial x}-\left(\frac{K_w}{\delta_w}K\rho_w+\frac{K_o}{\delta_o}K\rho_o\right)g\cdot\sin\alpha \tag{3.19}$$

Substituting the value of V from equation (3.19) in (3.16), we get

$$P\left(\frac{\partial S_w}{\partial t}\right)+\frac{\partial}{\partial x}\left[\frac{K_w}{\delta_w}K\left(\frac{1}{2}\frac{\partial P_c}{\partial S_w}\frac{\partial S_w}{\partial x}+\rho_w\cdot g\cdot\sin\alpha\right)\right]=0 \tag{3.20}$$

Using (3.6) and (3.8), we get

$$P\left(\frac{\partial S_w}{\partial t}\right)-\frac{\beta}{\delta_w}K\frac{\partial^2 S_w}{\partial x^2}+3\frac{\rho_w}{\delta_w}K\cdot g\cdot\sin\alpha\,(S_w)^2\frac{\partial S_w}{\partial x}=0 \tag{3.21}$$

Rewriting equation (3.21),

$$P\left(\frac{\partial S_w}{\partial t}\right)-\sigma\frac{\partial^2 S_w}{\partial x^2}+\delta S_w^2\frac{\partial S_w}{\partial x}=0 \tag{3.22}$$

Where:

$$\sigma=\frac{\beta}{\delta_w}K$$
$$\delta=3\frac{\rho_w}{\delta_w}K\cdot g\cdot\sin\alpha$$

With boundary condition:

$$S_w(0,t)=0.1;\ \ t\geq 0$$
$$S_w(1,t)=1;\ t\geq 0$$
$$S_w(x,0)=0.05;\ 0<x<1$$

This is a second order linear partial differential equation of motion for the flow of two immiscible liquids in a homogeneous medium. By using explicit method [13]

$$\frac{\partial S_w}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} + o(k) \tag{3.23}$$

$$\frac{\partial S_w}{\partial x} = \frac{u_{i,j} - u_{i-1,j}}{h} + o(h) \tag{3.24}$$

$$\frac{\partial^2 S_w}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + o(h^2) \tag{3.25}$$

Consider, $P = 0.5, \sigma = 0.5, \delta = 0.5$
Substituting these values in (3.22)
We get,

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] - \frac{k}{h}[u_{i,j}]^2[u_{i,j} - u_{i-1,j}] \tag{3.26}$$

This expression represents the explicit scheme for the governing partial differential equation. The solution of the expression in the form of tables and graphs is obtained using MATLAB.

4. Conclusion:

Value of saturation of water S_w at different points of x and t are given in the following Table 1. The Fig 1 represents the graph of $S_w(x,t)$ vs t. The graph of $S_w(x,t)$ vs x is defined in the Fig 2. Now we can generalize the problem with further iterations and the solution becomes clearer. For that we used MATLAB programming and get the nearest solution of the problem. The 3D graph of value of saturation of water S_w at different points of x and t shows in Fig 3. This states that saturation of water S_w increasing when x and t increasing.

Table 1: The numerical value of phase saturation (S_w)

t	x=0	x=0.25	x=0.5	x=0.75	x=1
0	0.1	0.05	0.05	0.05	1
0.025	0.1	0.0700125	0.05	0.43	1
0.05	0.1	0.0740172	0.21001	0.4989738	1
0.075	0.1	0.13882168	0.27059861	0.57660429	1
0.1	0.1	0.17592897	0.33932519	0.61338646	1
0.125	0.1	0.21068086	0.38170984	0.64809601	1
0.15	0.1	0.23432883	0.41736078	0.67111416	1
0.175	0.1	0.25307248	0.44246112	0.68973824	1
0.2	0.1	0.26661858	0.46190882	0.70316816	1
0.225	0.1	0.27690283	0.47612975	0.7134682	1
0.25	0.1	0.28447606	0.4868579	0.72106414	1
0.275	0.1	0.29014547	0.49479059	0.72677882	1
0.3	0.1	0.2943446	0.50071776	0.73101821	1
0.325	0.1	0.29747225	0.50511452	0.73418377	1
0.35	0.1	0.29979283	0.50838751	0.73653514	1
0.375	0.1	0.30151792	0.5108174	0.73828539	1
0.4	0.1	0.30279848	0.51262346	0.73958555	1

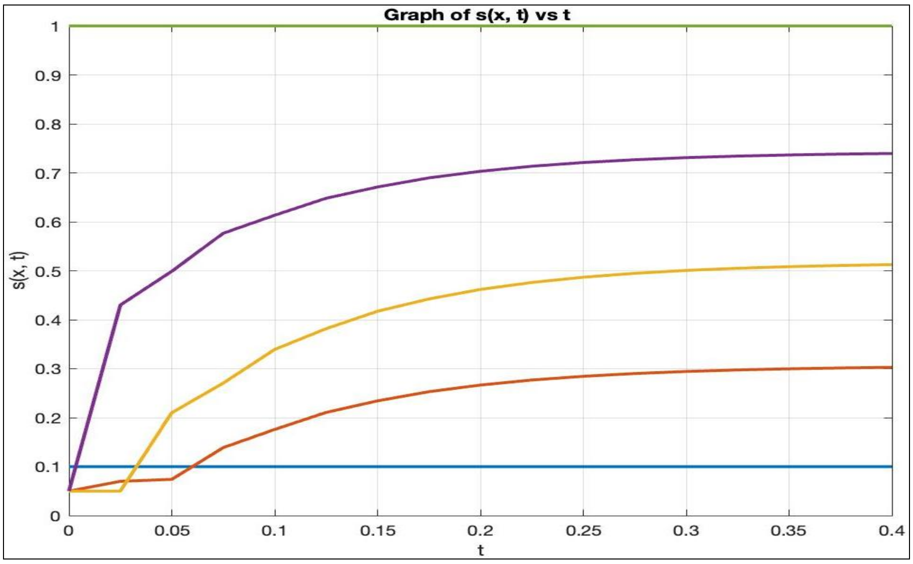

Fig 1: Graph of S(x,T) → T

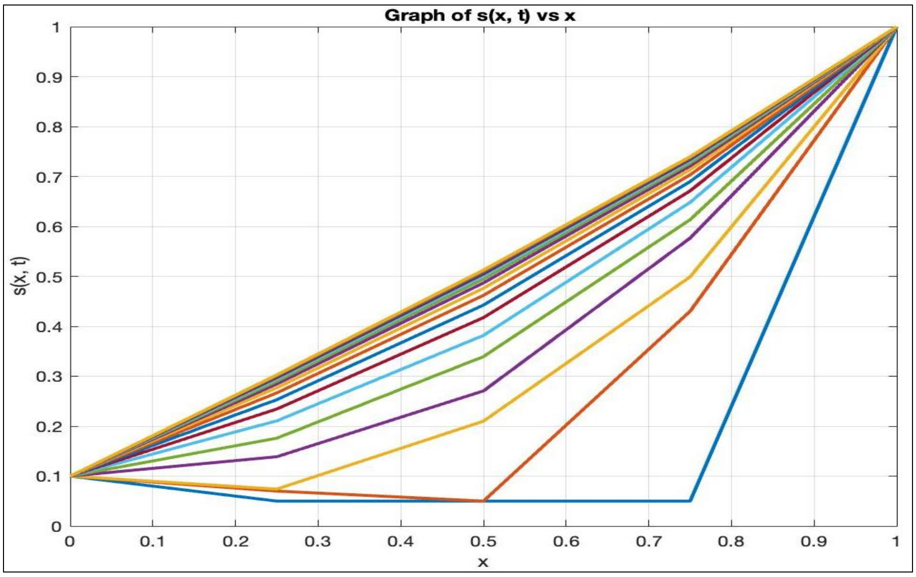

Fig 2: Graph of S(x,T) → x

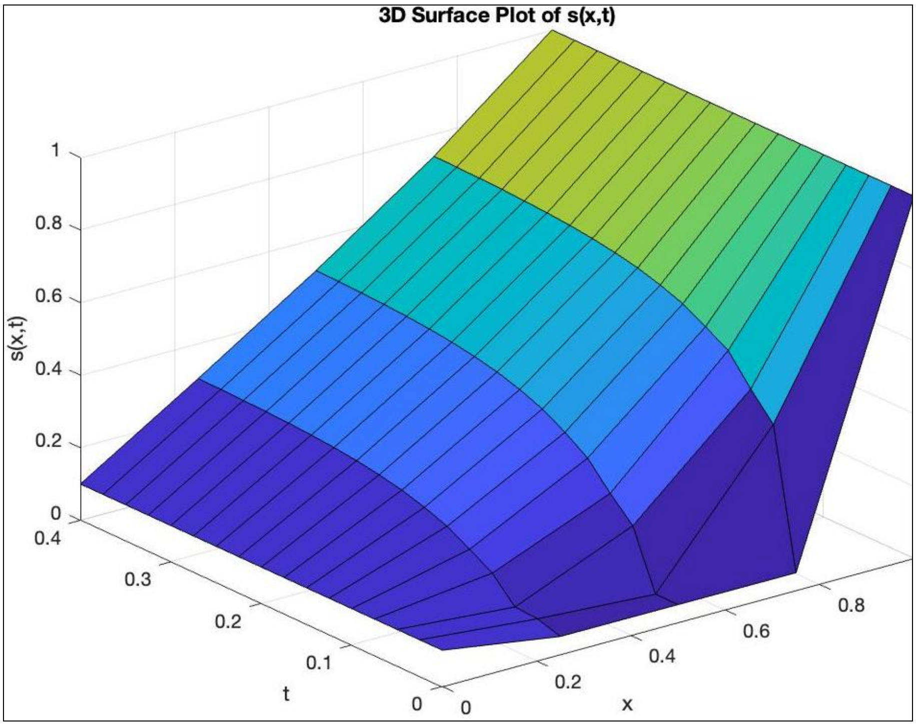

Fig 3: 3D Surface plot of S(x,t)

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