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Global Solution of a Non-Linear Kirchhoff – Carrier Type Equation with Dissipation

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Abstract
This paper demonstrates the existence and uniqueness of a local weak solution for the mixed problem (P). This problem is described in Equation (1). To assess the existence of global solutions, we used the Faedo-Galerkin method, the Aubin-Lions theorem of compactness, and important inequalities of functional analysis. Additionally, with respect to the uniqueness of the global solutions, we used the energy method due to the solution's regularity. We also used some inequalities of functional analysis. We use Nakao's lemma and some functional analysis inequalities to treat the asymptotic behavior of the problem.

Keywords: Existence and Uniqueness, Asymptotic Behavior, Global Weak Solution

1. Introduction
In this work, we study the existence and uniqueness of the decay of global solutions for the mixed problem, which is represented by (P).

$$(P)\left\|\begin{array}{l} u''(t) - M(|\nabla u(t)|^2)\Delta u(t) + M_1(|u(t)|^2)u(t) + \alpha u'(t) = f in Q \\ u(t) = 0 \; in \; \Sigma = \Gamma \times]0,T[\\ u(0) = u_0, \;\; u'(0) = u_1 \; in \; \Omega \end{array}\right. \tag{1}$$

Here, Ω denotes an open limited of R^n, where $n \geq 1$, and Ω has a smooth boundary $\partial\Omega = \Gamma$. $M \in C^1([0,\infty))$ is a positive function with additional hypothesis, and $\alpha \geq 0$. For each real fixed number, however arbitrary, $T > 0$. Q denotes the cylinder $Q = \Omega \times]0,T[$ with the lateral boundary $\sum = \Gamma \times]0,T[$. Furthermore, $-\Delta$ is the self-adjoint non limited operator that is defined by the set of elements $\{H_0^1(\Omega), L^2(\Omega), a(u,v)\}$, where

$$a(u,v) = \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}dx \tag{2}$$

2. Notation and hypothesis
In what follows, we use $((\;)), \|\;\|, (\;), |\;|$ to denote the inner product and the norm in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. Considering the space $H_0^1(\Omega)$ that is provided by the norm of gradient, if $u(t) \in H_0^1(\Omega)$, then $\|u(t)\| = |\nabla u(t)|$. Therefore, we assume the following hypotheses about M and M_1.

$$\begin{array}{l} H.1) M, M_1 \in C^1(0,\infty); R) \\ H.2) M(s) \geq m_0 > 0, \forall s \in 0,\infty) \\ H.3) M_1(s) \geq m_1 > 0, \forall s \in 0,\infty) \end{array} \tag{3}$$

3. Principal results
Consider the functions M and M_1 that satisfy hypotheses $H_1, H_2 and H_3$. If $u_0 \in H_0^1\,(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1\,(\Omega)$ and $f \in L^2(0,T; H_0^1(\Omega))$, then we have $T_0 > 0$, $0 < T_0 < T$ and a single vector function $u:]0,T[\to L^2(\Omega)$ such that

$\circ\ u(t) \in L^{\infty}(0, T_0; H_0^1(\Omega) \cap H^2(\Omega))$
$\circ\ u'(t) \in L^{\infty}(0, T_0; H_0^1(\Omega))$
$\circ\ u''(t) \in L^2(0, T_0; L^2(\Omega))$
$\circ\ \frac{d}{dt}(u'(t), v) + M(|\nabla u(t)|^2)(\Delta u(t), v) + M_1(|u(t)|^2)(u(t), v) +$ (4)
$+(\alpha u'(t), v) = (f(t), v)\, in \mathcal{D}'(0, T_0); \forall v \in H_0^1(\Omega)$
$\circ\ u(0) = u_0, u'(0) = u_1$

Comments

The existence of global solutions will be proven using the Faedo-Galerkin method, the compactness theorem of Aubin-Lions, and important inequalities of functional analysis. Further, the uniqueness of the global solutions will be proved using the energy method due the regularity of the weak solution and inequalities of functional analysis (Frota and Larkin, 1997)[6]. Meanwhile, with respect to the asymptotic behavior, we will use Nakao's method (Nakao, 2025)[5].

Approximate problem

We consider that $\{w_j\}_{j \in \mathrm{N}}$ is a complete orthonormal system of $L^2(\Omega)$ that is constituted by the eigenvector operator of $-\Delta$ and $\{\lambda_j\}_{j \in \mathrm{N}}$ is the corresponding sequence of eigenvalues. For each m=1,2,3,..., $V_m = \{w_1, w_2, w_3 \dots . w_m\}$ is the subspace that is created by $w_1, w_2, w_3 \dots . w_m$.

The approximate problem associated with (P) consists of finding a solution in the form

$$u_m(t) = g_{jm}(t)\omega_j(x) \in V_m \tag{5}$$

g_{jm} and the class C^2 are determined to satisfy the following system:

$$(PA)\left\| \begin{array}{l} (u_m''(t), v) - M_0(|\nabla u_m(t)|^2)(\Delta u_m(t), v) + M_1(|u_m(t)|^2)(u_m(t), v) + (\alpha u_m'(t), v) = (f(t), v) \\ u_m(0) = u_{0m} \to u_0 in H_0^1(\boldsymbol{\Omega}) \cap H^2(\boldsymbol{\Omega}) \\ u_m'(0) = u_{1m} \to u_1 in H_0^1(\Omega) \end{array} \right. \tag{6}$$

$\forall v \in V_m$ and $\forall j = 1,2, \dots \dots, m$

Here, u_{0m} and u_{1m} are the approaches of u_0 and u_1, respectively. Thus, here, $u_0 \in H_0^1(\boldsymbol{\Omega}) \cap H^2(\boldsymbol{\Omega})$ and $u_1 \in H_0^1(\Omega)$. Therefore, it is possible to approach finite linear combinations of ω_j, and there are constants α_{jm} and $\beta_{jm} \in \mathbb{R}$ where $j = 1,2, \dots . m$ such that

$$u_{0m} = \sum_{j=1}^{m} \alpha_{jm}\, \omega_j \to u_0 \text{strong in} \mathrm{H}_0^1(\boldsymbol{\Omega}) \cap H^2(\boldsymbol{\Omega})$$

and

$$u_{1m} = \sum_{j=1}^{m} \beta_{jm}\, \omega_j \to u_1 \text{strong in} \mathrm{H}_0^1(\Omega) \tag{7}$$

Then, $u_m(0) = u_{0m}$ and $u_m'(0) = u_{1m}$. Therefore, since there is a single linear combination of vectors V_m, it follows that

$$g_{jm}(0) = \alpha_{jm}, g_{jm}'(0) = \beta_{jm}\ (j = 1, 2, \dots, m) \tag{8}$$

Substituting $u_m(t)$ in (6) and using the fact of that $\{w_j\}_{j \in \mathrm{N}}$ is a complete orthonormal system in $L^2(\Omega)$, the following equation is obtained:

$$\left(\sum_{j=1}^{m} g_{jm}''(t)\omega_j(x), \omega_i(x)\right) - M(|\nabla u_m(t)|^2)\left(\Delta \sum_{j=1}^{m} g_{jm}(t)\omega_j(x), \omega_i(x)\right) +$$
$$+M_1(|u_m(t)|^2)\left(\sum_{j=1}^{m} g_{jm}(t)\omega_j(x), \omega_i(x)\right) + \left(\alpha \sum_{j=1}^{m} g_{jm}'(t)\omega_j(x), \omega_i(x)\right) = (f(t), \omega_i(x)) \tag{9}$$

We make some calculations and transform the system (PA) in one system $(\overline{PA})$ as follows.

$$(P\bar{A})\left\| \begin{array}{l} g_{jm}'' - \lambda_j M(|\nabla u_m(t)|^2) g_{jm}(t) + M_1(|u_m(t)|^2) g_{jm}(t) + \alpha g_{jm}'(t) = (f(t), \omega_j(x)) \\ g_{jm}(0) = \alpha_{jm}, g_{jm}'(0) = \beta_{jm} (j = 1,2, \dots, m) \lambda_j = \lambda \end{array} \right. \tag{10}$$

Matrix form

To simplify the system in (10), we write it in the matrix form using $\lambda_j = \lambda$ and

$$X = \begin{bmatrix} g_{1m} \\ g_{2m} \\ \& \quad \vdots \\ g'_{mm} \end{bmatrix} \tag{11}$$

We then obtained $(\overline{PA})_1$ using

$$\begin{bmatrix} g' \\ \\ g'' \end{bmatrix} = \begin{bmatrix} 01 \\ \\ \lambda M_0(|\nabla u_m(t)|^2) - M_1(|u_m(t)|^2)\alpha \end{bmatrix} \begin{bmatrix} g \\ \\ g' \end{bmatrix} + \begin{bmatrix} (f, \omega_1) \\ \\ (f, \omega_m) \end{bmatrix} \tag{12}$$

Thus,

$$X'' = AX + B \tag{13}$$

We reorganize it into this form:

$$Y = \begin{bmatrix} X \\ X' \end{bmatrix}, Y' = \begin{bmatrix} X' \\ X' \end{bmatrix} \tag{14}$$

We obtain the form

$$\begin{bmatrix} X' \\ AX + B \end{bmatrix}_{2m\times 1} = \begin{bmatrix} 0_{m\times m} I_{m\times m} \\ A_{m\times m} 0_{m\times m} \end{bmatrix}_{2m\times 2m} \begin{bmatrix} X \\ X' \end{bmatrix}_{2m\times 1} + \begin{bmatrix} 0_{m\times 1} \\ B_{m\times 1} \end{bmatrix}_{2m\times 1} \tag{15}$$

Thus, the system $(P\bar{A})$ can be rewritten in following form:

$$\left\| \begin{array}{l} Y' = DY + C = F(t, Y) \\ Y(0) = \begin{bmatrix} X(0) \\ X'(0) \end{bmatrix} \end{array} \right. \tag{16}$$

Next,

$$X'' = AX + B, D = \begin{bmatrix} 0_{m\times m} I_{m\times m} \\ A_{m\times m} 0_{m\times m} \end{bmatrix}_{2m\times 2m}, Y = \begin{bmatrix} X \\ X' \end{bmatrix}_{2m\times 1}, C = \begin{bmatrix} 0_{m\times 1} \\ B_{m\times 1} \end{bmatrix}_{2m\times 1} \tag{17}$$

We verify that the above system satisfies the conditions of Caratheodory's theorem (Brézis, 1984)[1]. Therefore, it has a solution $u_m(t)$ that is defined by $0 < t_m \leq T_0$.

The a priori estimate implies that $u_m(t)$ is defined in the interval $[0, T_0]$.

A priori estimate

First Estimate

Considering $v = 2u_m'$ in the system$(PA)_1$, we obtain the following result:

$$|u'_m(t)|^2 + \|u_m(t)\|^2 \leq \frac{k}{k_2} + \frac{k_1}{k_2}\int_0^t |u'_m(s)|^2 \, ds \tag{18}$$

By applying the Gronwall's inequality (Lions, 1969)[2],

$$|u'_m(t)|^2 + \|u_m(t)\|^2 \leq c(cte) \tag{19}$$

It follows that

$$|u'_m(t)| \leq c_1, \|u_m(t)\| \leq c_2, \forall m, \forall t \in 0, t_m), \tag{20}$$

Applying Caratheodory's theorem can extend the solution $u_m(t)$ to the interval $[0, T]$. Thus, the sequences can be expressed as follows:

$$(u_m)_{m\in\mathbb{N}} \text{is bounded in} L^\infty(0, T; H_0^1(\Omega)) \quad (21)$$

$$(u_m')_{m\in\mathbb{N}} \text{is bounded in } L^\infty(0, T; L^2(\Omega)) \quad (22)$$

Second Estimate
Consider $v = -2\Delta u_m'(t)$ in the system$(PA)_1$, we obtain the following result:

$$\|u_m'(t)\|^2 + |\Delta u_m(t)|^2 \le \frac{k_1}{k_5} + \frac{k_4}{k_5}\int_0^t \left[\varphi(s) + \frac{\varphi(s)^2}{2}\right] ds \quad (23)$$

Here,

$$\varphi(t) = \|u_m'(t)\|^2 + |\Delta u_m(t)|^2 \quad (24)$$

The inequality (23) can be rewritten in the following form:

$$\varphi(t) \le \frac{k_1}{k_5} + \frac{k_4}{k_5}\int_0^t \left[\varphi(s) + \frac{\varphi(s)^2}{2}\right] ds \quad (25)$$

Using Gronwall's inequality, it follows that

$$\varphi(t) \le k \text{ where, } \|u_m'(t)\|^2 + |\Delta u_m(t)|^2 \le k(cte) \quad (26)$$

We observe that $\varphi(t)$ is continuous in the interval $[0, T_0]$. Then, $T_0 < T$ exists such that $\varphi(t) \le k$ for all m and all $t \in [0, T_0]$. From the inequality (23), it follows that

$$\|u_m'(t)\| \le k, \forall m, \forall t \in [0, T_0] \quad (27)$$

$$|\Delta u_m(t)| \le k, \forall m, \forall t \in [0, T_0] \quad (28)$$

Thus,

$$(u_m') \text{ is bounded in } L^\infty(0, T_0; H_0^1(\Omega)) \quad (29)$$

$$(\Delta u_m) \text{ is bounded in } L^\infty(0, T_0; L^2(\Omega)) \quad (30)$$

Following the first and second estimate (u_m) and (Δu_m), we obtain that (u_m) is bounded in

$$L^\infty(0, T_0; H_0^1(\Omega) \cap H^2(\Omega)) \quad (31)$$

Third Estimate
Considering $v = 2u_m''(t)$ in the system $(PA)_1$, using the Cauchy-Schwartz inequalities and the elementary inequality Rivera, 2004) [3], and integrating from 0 to t, we obtain the following:

$$2\int_0^t |u_m''(s)|^2 ds + |u_m'(s)|^2 \le k + \frac{3}{2}\int_0^t |u_m''(s)|^2 ds \quad (32)$$

It follows that

$$\int_0^t |u_m''(s)|^2 ds \le k \quad (33)$$

Therefore,

$$(u_m'') \text{ is bounded in } L^2(0, T_0; L^2(\Omega)) \quad (34)$$

Passage to the limit
The previous estimates can be expressed as follows.

$(u_m(t))$ is bounded in $L^\infty(0,T_0;H_0^1(\Omega)\cap H^2(\Omega))$ (35)

$(u_m'(t))$ is bounded in $L^\infty(0,T_0;H_0^1(\Omega))$ (36)

$(u_m''(t))$ is bounded in $L^2(0,T_0;L^2(\Omega))$ (37)

Using the corollary of Banach-Alaoglu-Bourbaki (Carrier, 1945)[4] and observing that $L^2(0,T_0;L^2(\Omega))$ is Hilbert, the following is obtained:

$$u_m \to u \text{ in } L^\infty(0,T_0;H_0^1(\Omega)\cap H^2(\Omega)), \text{ weak}^* \quad (38)$$

$$\Leftrightarrow$$

$$u_m \to u \text{ in } L^\infty(0,T_0;H_0^1(\Omega)\cap H^2(\Omega)), \text{ weak} \quad (39)$$

$$\Leftrightarrow$$

$$(\Delta u_m,\omega) \to (\Delta u,\omega), \forall\omega \in L^2(0,T_0;L^2(\Omega)), \text{ weak} \quad (40)$$

We also have

$$u_m' \to u' \text{ in } L^\infty(0,T_0;H_0^1(\Omega)), \text{ weak}^* \quad (41)$$

$$\alpha u_m' \to \alpha u' \text{ in } L^\infty(0,T_0;H_0^1(\Omega)), \text{ weak}^* \quad (42)$$

Conversely, we can write

$$u_m'' \to u'' \text{ in } L^2(0,T_0;L^2(\Omega)), \text{ weak} \quad (43)$$

$$\Leftrightarrow$$

$$(u_m'',\omega) \to (u'',\omega), \forall\omega \in L^2(0,T_0;L^2(\Omega)), \text{ weak} \quad (44)$$

Convergences of M and M_1

According to the Aubin-Lions lemma on compactness (Hosoya and Yamada, 1991)[7], taking $B_0 = H_0^1(\Omega)\cap H^2(\Omega)$ and $B = B_1 = H_0^1(\Omega)$, implies the following:

$$u_m \to u \text{ strong in } L^2(0,T_0;H_0^1(\Omega)) \quad (45)$$

Using some results of functional analysis leads to the following convergence:

$$M(\|u_m(t)\|^2) \to M(\|u(t)\|^2), \text{ in } [0,T_0] \quad (46)$$

Hence, it can be concluded that

$$M(\|u_m(t)\|^2)((u_m(t),v)) \to M(\|u(t)\|^2)((u(t),v)) \text{ in } L^2(0,T_0;H_0^1(\Omega)) \quad (47)$$

For the convergence of another nonlinear term, knowing that $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$, we have

$$M_1(|u_m(t)|^2)(u_m(t),v) \to M_1(|u(t)|^2)(u(t),v) \text{ in } L^2(0,T_0;L^2(\Omega)) \quad (48)$$

By multiplying $(PA)_1$ by $\theta \in D(0,T_0)$ and integrating 0 to T_0 as $\theta(t)v \in L^2(\Omega)$, according to previous convergences, when $m \to \infty$, we obtain

$$\int_0^{T_0}(\alpha u_m'(t),\theta(t)v)\,dt \to \int_0^{T_0}(\alpha u'(t),\theta(t)v)\,dt \quad (49)$$

$$\int_0^{T_0}(u_m''(t),\theta(t)v)\,dt \to \int_0^{T_0}(u''(t),\theta(t)v)\,dt \quad (50)$$

$$\int_0^{T_0}(M(\|u_m(t)\|^2)\Delta u_m(t),\theta(t)v)\,dt \to \int_0^{T_0}(M(\|u(t)\|^2)\Delta u(t),\theta(t)v)\,dt \quad (51)$$

$$\int_0^{T_0}(M_1(\|u_m(t)\|^2)\Delta u_m(t),\theta(t)v)\,dt \to \int_0^{T_0}(M_1(|u(t)|^2)u(t),\theta(t)v)\,dt \quad (52)$$

Thus, by taking the limit when $m \to \infty$, using $v \in V_m$, and holding *m* arbitrary fixed, we can express the following:

$$\int_0^{T_0}(u''(t) - M(\|u(t)\|^2)\Delta u(t) + M_1(|u(t)|^2)u(t) + \alpha u'(t) - f(t), v\theta(t))\,dt = 0 \tag{53}$$

Knowing that the union of V_m is dense in $L^2(\Omega)$ and the above equality above is valid when $\forall v \in V_m$ and $m_0 < \infty$, it follows that the equality is equal to $v \in L^2(\Omega)$. Furthermore, the set $\{v\theta; v \in L^2(\Omega); \theta \in L^2(0,T_0)\}$ is dense in $L^2(\Omega)$. Then,

$$\int_0^{T_0}(u''(t) - M(\|u(t)\|^2)\Delta u(t) + M_1(|u(t)|^2)u(t) + \alpha u'(t) - f(t), \omega)\,dt = 0, \forall \omega \in L^2(\Omega) \tag{54}$$

Finally,

$$\omega = u''(t) - M(\|u(t)\|^2)\Delta u(t) + M_1(|u(t)|^2)u(t) + \alpha u'(t) - f(t). \text{ Then, } \|\omega\|_{L^2(\Omega)} = 0 \tag{55}$$

Therefore,

$$\omega = 0 in \Omega \tag{56}$$

This completes the proof.

Initial conditions

This section proves that $u(0) = u_0$ and $u'(0) = u_1$ through the following demonstration. From previous results, we know the following:

$$u \in L^2(0,T_0; H_0^1(\Omega) \cap H^2(\Omega)) \tag{57}$$

$$u' \in L^2(0,T_0; H_0^1(\Omega)) \tag{58}$$

$$u'' \in L^2(0,T_0; L^2(\Omega)) \tag{59}$$

Using the results of the regularity, it follows that

$$\left| \begin{array}{l} u \in C^0(0,T_0); H_0^1(\Omega)) \\ u' \in C^0(0,T_0); L^2(\Omega)) \end{array} \right. \tag{60}$$

Thus, it can be concluded that $u \in C^1([0,T_0], L^2(\Omega))$. Therefore, it makes sense to calculate $u(0)$ and $u'(0)$. Consider $\theta \in C^1([0,T_0])$, $\theta(0) = 1$, $\theta(T_0) = 0$ and $v \in H_0^1(\Omega)$. First, we calculate $u(0) = u_0$. We know that

$$u_m' \to u' \text{ in } L^2(0,T_0; H_0^1(\Omega)) \tag{61}$$

and

$$((u_m', \omega)) \to ((u', \omega)), \forall \omega \in L^2(0,T_0; H_0^1(\Omega)) \tag{62}$$

We take $\omega(t) = \theta(t)v$ with $\theta(t) \in L^2(0,T_0)$ and $v \in H_0^1(\Omega)$ and integrate 0 to T_0. After integrating it by parts, we obtain

$$-((u_m(0), v)) - \int_0^{T_0}((u_m(t), v))\,\theta'(t)dt \to -((u(0), v)) - \int_0^{T_0}((u(t), v))\,\theta'(t)dt \tag{63}$$

From the convergence in (38), the following result is obtained:

$$\int_0^{T_0}((u_m(t), v))\,\phi dt \to \int_0^{T_0}((u(t), v))\,\phi dt, \forall v \in H_0^1(\Omega) \text{and} \forall \phi \in L^1(\Omega) \tag{64}$$

Lebesgue's dominated convergence is used to obtain the following:

$$((u_m(0), v)) \to ((u(0), v)) \tag{65}$$

It follows that

$$u_m(0) = u_{0m} \to u_0 \text{in} H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow H_0^1(\Omega) \tag{66}$$

Then,

$$((u_m(0), v)) \to ((u_0, v)), \forall v \in H_0^1(\Omega) \tag{67}$$

Therefore,

$$u(0) = u_0 \tag{68}$$

To show that $u^{'}(0) = u_1$, we us the convergences in (39)–(40). Analogously, we obtain

$$((u_m^{'}(0), v)) \to ((u_1, v)) \forall v \in L^2(\Omega) \tag{69}$$

Therefore,

$$u^{'}(0) = u_1 \tag{70}$$

In the next section, we will prove the uniqueness of solution to achieve our goal.

4. Uniqueness of solution

Theorem 1 (Uniqueness): The problem (P) has a unique solution

Suppose that *u* and *w* are two vector functions that are defined using $[0, T_0]$ in $L^2(\Omega)$ such that they are solutions of (P) according to the conditions of the main Theorem 1. Considering $r(t) = u(t) - \omega(t)$, we obtain the following:

$$r \in L^\infty\big(0, T_0; H_0^1(\Omega) \cap H^2(\Omega)\big) \tag{71}$$

$$r^{'} \in L^\infty\big(0, T_0; H_0^1(\Omega)\big) \tag{72}$$

$$r^{''} \in L^\infty\big(0, T_0; L^2(\Omega)\big) \tag{73}$$

Considering that $\{u(t), \omega(t)\}$ are solutions to the problem (P), then

$$u^{''}(t) - M(\|u(t)\|^2)\Delta u(t) + M_1(|u(t)|^2)u(t) + \alpha u^{'}(t) = f \tag{74}$$

$$\omega^{''}(t) - M(\|\omega(t)\|^2)\Delta\omega(t) + M_1(|\omega(t)|^2)\omega(t) + \alpha\omega^{'}(t) = f \tag{75}$$

Adding and subtracting the terms $M(\|u(t)\|^2)\Delta\omega(t)$ and $M_1(|u(t)|^2)\omega(t)$ in (74)–(75) implies that

$$\begin{aligned} &r^{''}(t) - M(\|u(t)\|^2)\Delta r(t) - \big(M(\|u(t)\|^2) - M(\|\omega(t)\|^2)\big)\Delta\omega(t) + M_1(|u(t)|^2)r(t) + \\ &+\big(M_1(|u(t)|^2) - M_1(|\omega(t)|^2)\big)\omega(t) + \alpha r^{'}(t) = 0 \end{aligned} \tag{76}$$

Because $M_i \in C^1 0, \infty)$, using Mean Value Theorem, according to the regularity of the solution that is obtained, and associating with $2r^{'}(t)$ in $L^2(0, T_0; H_0^1(\Omega))$ in (76), the following is defined:

$$\begin{aligned} &\frac{d}{dt}\left|r^{'}(t)\right|^2 + M(\|u(t)\|^2)\frac{d}{dt}\|r(t)\|^2 + M_1(|u(t)|^2)\frac{d}{dt}|r(t)|^2 + 2\alpha\left\|r^{'}(t)\right\|^2 = \\ &2M^{'}(\xi_1)[\|u(t)\|^2 - \|\omega(t)\|^2](\Delta\omega(t), r^{'}(t)) - 2M_1^{'}(\xi_2)[|u(t)|^2 - |\omega(t)|^2](\omega(t), r^{'}(t)) \end{aligned} \tag{77}$$

By Using the inequality of Cauchy - Schwarz in the second part of (77), we obtain

$$\begin{aligned} &\frac{d}{dt}\left|r^{'}(t)\right|^2 + M(\|u(t)\|^2)\frac{d}{dt}\|r(t)\|^2 + M_1(|u(t)|^2)\frac{d}{dt}|r(t)|^2 + 2\alpha\left\|r^{'}(t)\right\|^2 \le \\ &2\left|M^{'}(\xi_1)\right|\left|\|u(t)\|^2 - \|\omega(t)\|^2\right||\Delta\omega(t)|\left|r^{'}(t)\right| + 2\left|M_1^{'}(\xi_2)\right|\left||u(t)|^2 - |\omega(t)|^2\right||\omega(t)|\left|r^{'}(t)\right| \end{aligned} \tag{78}$$

We note that

$$\begin{aligned} &\left|\|u(t)\|^2 - \|\omega(t)\|^2\right| = \left|\|u(t)\| + \|\omega(t)\|\right|\left|\|u(t)\| - \|\omega(t)\|\right| \le k_1\|r(t)\| \\ &\left||u(t)|^2 - |\omega(t)|^2\right| = \left||u(t)| + |\omega(t)|\right|\left||u(t)| - |\omega(t)|\right| \le k_2|r(t)| \end{aligned} \tag{79}$$

Substituting (79) in (78) and using the definition $M_i \in C^1 0, \infty)$, we obtain

$$\frac{d}{dt}\left|r^{'}(t)\right|^{2} + M(\|u(t)\|^{2})\frac{d}{dt}\|r(t)\|^{2} + M_1(|u(t)|^{2})\frac{d}{dt}|r(t)|^{2} + 2\alpha\left\|r^{'}(t)\right\|^{2} \leq 2c_0k_1\|r(t)\|\left|r^{'}(t)\right| + 2c_1k_2|r(t)|\left|r^{'}(t)\right| \quad (80)$$

Where

$$\left|M^{'}(\xi_1)\right||\Delta\omega\ (t)| \leq c_0 \text{ and } \left|M_1^{'}(\xi_2)\right||\Delta\omega\ (t)| \leq c_1 \quad (81)$$

By applying the elementary inequality $2ab \leq a^2 + b^2$ and using $c_3 = c_0k_1 + c_1k_2$ in the second part of (80), we obtain

$$\frac{d}{dt}\left[\left|r^{'}(t)\right|^{2} + M(\|u(t)\|^{2})\|r(t)\|^{2} + M_1(|u(t)|^{2})|r(t)|^{2}\right] + 2\alpha\underbrace{\left\|r^{'}(t)\right\|^{2}}_{\geq 0} \leq c_3\left[\|r(t)\|^{2} + |r(t)|^{2} + \left|r^{'}(t)\right|^{2}\right] + \\ +\left|M^{'}(\|u(t)\|^{2})\right|2\left|\left((u(t),u^{'}(t))\right)\right|\|r(t)\|^{2} + \left|M_1^{'}(|u(t)|^{2})\right|2\left|(u(t),u^{'}(t))\right||r(t)|^{2} \quad (82)$$

We again use the Cauchy-Schwartz inequality in (82) and observe the limitations of the terms for

$$\|u(t)\|^{2}, |u(t)|^{2}, \left\|u^{'}(t)\right\|^{2}, \left|M_0^{'}(\|u(t)\|^{2})\right| \text{ and } \left|M_1^{'}(\|u(t)\|^{2})\right| \quad (83)$$

Where

$$\alpha_1 = \left|M^{'}(\|u(t)\|)\right|2\|u(t)\|\left\|u^{'}(t)\right\| \\ \alpha_2 = \left|M_1^{'}(|u(t)|)\right|2|u(t)|\left|u^{'}(t)\right| \\ c_4 = max\{c_3, 2\alpha_1, 2\alpha_2\} \quad (84)$$

Hence,

$$\frac{d}{dt}\left[\left|r^{'}(t)\right|^{2} + M(\|u(t)\|^{2})\|r(t)\|^{2} + M_1(|u(t)|^{2})|r(t)|^{2}\right] \leq c_4\left[\|r(t)\|^{2} + |r(t)|^{2} + \left|r^{'}(t)\right|^{2}\right] \quad (85)$$

By integrating the inequality in (85) from 0 to t and using the hypothesis of $M_i \in C^1 0,\infty)$, the following is obtained.

$$\left|r^{'}(t)\right|^{2} + m_0\|r(t)\|^{2} + m_1|r(t)|^{2} \leq c_5\int_0^t\left[\|r(s)\|^{2} + |r(s)|^{2} + \left|r^{'}(s)\right|^{2}\right]ds + c_7 \quad (86)$$

By substituting $m_2 = min\{1, m_0, m_1,\}$ into inequality (86) and making some algebraic manipulations, we obtain the following:

$$\left|r^{'}(t)\right|^{2} + \|r(t)\|^{2} + |r(t)|^{2} \leq c_7 + c_5\int_0^t\left[\|r(s)\|^{2} + |r(s)|^{2} + \left|r^{'}(s)\right|^{2}\right]ds \quad (87)$$

Conversely, using Gronwall's inequality in (87), we deduce that

$$\left|r^{'}(t)\right|^{2} + \|r(t)\|^{2} + |r(t)|^{2} = 0 \quad (88)$$

This implies that $|r(t)| = 0$, and thus $r(t) = 0$. Therefore,

$$u(t) = \omega(t), \forall t \in [0, T_0] \quad (89)$$

5. Exponential decay

Theorem 2 (Exponential decay)

The solution of the problem (P) decays exponentially when $t \to \infty$. Then, there are positive constants $\delta > 0$and $\lambda > 0$ such that

$$E(t) \leq \lambda E(0)e^{-\delta t}, \text{ where} E(t) = \frac{1}{2}\left[\left|u^{'}(t)\right|^{2} + \widehat{M}(\|u(t)\|^{2}) + \widehat{M_1}(|u(t)|^{2})\right] \quad (90)$$

Where E (t) is the energy associated with the problem (P) if u is a global solution of the problem (P).

5. 1. 1. Demonstration:

We compose the first equation of the problem (P) that is shown above using$v = u^{'}(t)$.

$$(u^{''}(t), u^{'}(t)) - M(\|u(t)\|^{2})(\Delta u(t), u^{'}(t)) + M_1(|u(t)|^{2})(u(t), u^{'}(t)) + (\alpha u^{'}(t), u^{'}(t)) = 0 \quad (91)$$

Now, if we consider

$$\widehat{M_i}(\lambda) = \int_0^{\lambda} M_i(s)ds \quad (92)$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\left[\left|u'(t)\right|^2 + \widehat{M}(\|u(t)\|^2) + \widehat{M_1}(|u(t)|^2)\right] + \alpha\left|u'(t)\right|^2 = 0 \quad (93)$$

Finally,

$$\frac{1}{2}\frac{d}{dt}E(t) + \alpha\left|u'(t)\right|^2 = 0 \quad (94)$$

This means that the energy of the problem (P) is decreasing. By integrating (94) from 0 to t, the following equation is obtained:

$$E(t) + \alpha\int_0^t\left|u'(t)\right|^2 ds = E(0) \quad (95)$$

By integrating (94) from t_1 to t2 with $0 < t_1 < t_2$, the following equation is obtained:

$$E(t_2) + \alpha\int_{t_1}^{t_2}\left|u'(t)\right|^2 ds = E(t_1)\ \forall t > 0 \quad (96)$$

Therefore,

$$\int_t^{t+1}\left|u'(t)\right|^2 ds = \alpha\big(E(t) - E(t+1)\big) = \alpha F(t)^2 \quad (97)$$

Using the mean value theorem for integrals, we can observe that there exists two points

$$t_1 \in \left[t, t+\frac{1}{4}\right] \text{ and } t_2 \in \left[t+\frac{3}{4}, t+1\right] \quad (98)$$

such that

$$\frac{1}{4}\left|u'(t_1)\right|^2 = \int_t^{t+\frac{1}{4}}\left|u'(t_1)\right|^2 \le \alpha F(t)^2 \Rightarrow \left|u'(t_1)\right|^2 \le \alpha 4F(t)^2 \quad (99)$$

and

$$\frac{1}{4}\left|u'(t_2)\right|^2 = \int_{t+\frac{3}{4}}^{t+1}\left|u'(t_2)\right|^2 \le \alpha F(t)^2 \Rightarrow \left|u'(t_2)\right|^2 \le \alpha 4F(t)^2 \quad (100)$$

It follows that

$$\left\|u'(t_i)\right\|^2 \le \alpha 4F(t)^2 \Rightarrow \left\|u'(t_i)\right\| \le \sqrt{\alpha}2F(t),\ i = 1{,}2 \quad (101)$$

By using $v = u(t)$ in the first equation of the problem (P) and integrating from t_1 to t_2, it gives the following:

$$\int_{t_1}^{t_2} M(\|u(t)\|^2)\,\|u(t)\|^2 ds + \int_{t_1}^{t_2} M_1(|u(t)|^2)\,|u(t)|^2 ds = (u'(t_1), u(t_1)) - (u'(t_2), u(t_2)) - \alpha\int_{t_1}^{t_2}(u'(t), u(t))\,ds \quad (102)$$

By using the Cauchy-Schwartz and Elementary inequalities, it follows that

$$\int_{t_1}^{t_2} M\left(\|u(t)\|^2\right)\|u(t)\|^2 ds + \int_{t_1}^{t_2} M_1\left(|u(t)|^2\right)|u(t)|^2 ds \le \sup_{t<s<t+1} ess\left\{|u(s)|\left[|u'(t_1)|\right] + |u'(t_2)|\right\} + \\ + \frac{1}{2}\alpha\int_{t_1}^{t_2}|u'(t)|^2 ds + \frac{1}{2}\alpha\int_{t_1}^{t_2}|u(t)|^2 ds \quad (103)$$

Noting the immersion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $\left|u'\right|_{L^2}^2 \le c_1\left\|u'\right\|_{H_0^1}^2$, it follows that

$$\int_{t_1}^{t_2} M\left(\|u(t)\|^2\right)\|u(t)\|^2\, ds + \int_{t_1}^{t_2} M_1\left(|u(t)|^2\right)|u(t)|^2\, ds \le 4c_1 \sup_{t<s<t+1} ess\left[|u(s)|\right]\alpha F(t) +$$
$$+\left(\frac{1}{2}c_1\alpha\right)\int_{t_1}^{t_2}\|u'(t)\|^2 ds + \frac{1}{2}\alpha\int_{t_1}^{t_2}|u(t)|^2 ds \tag{104}$$

By using the hypothesis about M and M_1 and grouping the terms, we obtain the following:

$$\int_{t_1}^{t_2} m_0\|u(t)\|^2\, ds + \int_{t_1}^{t_2}\left(m_1 - \frac{1}{2}\alpha\right)|u(t)|^2\, ds \le 4c_1 \sup_{t<s<t+1} ess\left[|u(s)|\right]\alpha F(t) + \left(\frac{1}{2}c_1\alpha\right)\int_{t_1}^{t_2}\|u'(t)\|^2 ds \tag{105}$$

By using $m_1 > \frac{1}{2}$, we have

$$k\int_{t_1}^{t_2}\left[\|u(t)\|^2 + |u(t)|^2\right]ds \le 4c_1\alpha \sup_{t\le s\le t+1} ess\left[|u(s)|\right]F(t) + \left(\frac{1}{2}c_1\alpha\right)F(t)^2 \tag{106}$$

Here, $k = min\left\{m_0, \left(m_1 - \frac{1}{2}\alpha\right)\right\}$. By again using the immersion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and making some algebraic manipulations, we can obtain

$$\sup_{t\le s\le t+1} ess\, E(s) \le E(t^*) + \alpha\int_{t_1}^{t+1}|u'|^2\, ds \tag{107}$$

Then,

$$\sup_{t\le s\le t+1} ess\, E(s) \le c_6 F(t)^2 + \frac{1}{\delta m_1}\sup_{t\le s\le t+1} ress\, E(s) \tag{108}$$

Finally, it is concluded that

$$\left(1 - \frac{1}{\delta m_1}\right)\sup_{t\le s\le t+1} ess\, E(s) \le c_6 F(t)^2 \text{, where } m_1 \ge \frac{1}{\delta} \tag{109}$$

Conversely, we can observe that

$$\sup_{t\le s\le t+1} ess\, E(s) \le c_7 F(t)^2 = c_7\left(E(t) - E(t+1)\right) \tag{110}$$

Where

$$c_7 = \frac{c_6}{\left(1 - \frac{1}{\delta m_1}\right)} = \frac{c_6}{\frac{\delta m_1 - 1}{\delta m_1}} = \frac{c_6 \delta m_1}{\delta m_1 - 1} > 0 \tag{111}$$

Using Nakao's method, we conclude the following:

$$E(t) \le \lambda E(0)e^{-\delta t}, \forall t > 0 \tag{112}$$

where λ and $\delta$are positive constants.

6. Conclusion

The presented methodology was able to propose a global solution for the Non-Linear Kirchhoff – Carrier Type Equation with Dissipation.

7. References

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