**Int. j. adv. multidisc. res. stud.** 2025; 5(3):1475-1479

**Received:** 01-05-2025
**Accepted:** 10-06-2025

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

# Fixed Point Theorems in Fuzzy Metric Space by Rational Contraction Mapping

[1] **Shivani Chourasiya,** [2] **Kavita Shrivastava**
[1, 2] Department of Mathematics and Statistics, Dr. Harisingh Gour Vishwavidyalaya, Sagar M.P. - 470003, India

**DOI:** https://doi.org/10.62225/2583049X.2025.5.3.4475
Corresponding Author: **Shivani Chourasiya**

**Abstract**
Al-Thagafi and Shahzad introduced occasionally weakly compatible mapping in 2008. In 2011, S. Chauhan and S. Kumar developed a common fixed-point theorem for occasionally weakly compatible mapping for four single values self-maps in fuzzy metric space without addressing space completeness. Our paper generalizes S. Chauhan and S. Kumar applying eight self-mappings of a complete fuzzy metric space in rational contraction.

**Keywords:** Fixed Point, Fuzzy Metric Space, Weakly Compatible Map, Occasionally Weakly Compatible Map

**2010 Mathematics Subject Classification:** 47H10, 15B15

## 1. Introduction

The most comprehensive subfield of the nonlinear functional analysis is known as fixed point theory. Stefen Banach [2], a Polish mathematician, presented the contraction principle in 1922, which he used to prove the existence and uniqueness of the fixed-point theorem in metric spaces. This brought about a significant change in the field of mathematics, and other scholars are currently working on this topic in a variety of different methods (see, [10], [14], [18], [4] etc.) First and foremost, in the year 1965, L. A. Zadeh [20] presented us with the fuzzy set. The fuzzy metric spaces were subsequently established by Kramosil and Michalek [11] in the year 1975. Osmo Kaleva, Seppo Seikkala [8] in 1984 proved the fixed point theorem in fuzzy metric spaces. This was generalized in a different approach by Veeramani and George [5] in the year 1994. Gregori and Sapena [6] generalized George, Veeramani, Kramosil, and Michalek fuzzy metric spaces in 2002. Various writers have introduced fuzzy metric spaces in multiple methods (see, [9], [16], [12] etc.). Al-Thagafi and Shahzad [1] introduced the notion of occasionally weakly compatible mapping in 2008. Then, in 2011, S. Chauhan and S. Kumar [3] proved a common fixed point theorem for occasionally weakly compatible mapping for four single values self-maps in fuzzy metric space without considering space completeness. Our paper generalizes S. Chauhan and S. Kumar [3] by using eight self-mappings of a complete fuzzy metric space in rational contraction.

**Definition 1.1:** [15] A triangular norm or t-norm $*$ is a binary operation on the unit interval [0, 1] such that for all $p, q, r, s \in [0, 1]$ and the following conditions satisfied:

(1) $p * 1 = p$
(2) $p * q = q * p$
(3) $p * q \leq r * s$ whenever $p \leq r$ and $q \leq s$
(4) $(p * q) * r = p * (q * r)$

**Definition 1.2:** [5] The three tuple $(\Omega, M, *)$ is said to be a fuzzy metric space if $\Omega$ is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $\Omega\times\Omega\times(0, \infty)$ satisfying the following conditions:
(1) $M(\mu, \upsilon, t) > 0$,
(2) $M(\mu, \upsilon, t) = 1 \Leftrightarrow \mu = \upsilon$,
(3) $M(\mu, \upsilon, t) = M(\upsilon, \mu, t)$,
(4) $M(\mu, \upsilon, t) * M(\upsilon, \theta, s) \leq M(\mu, \theta, t + s)$
(5) $M(\mu, \upsilon, .): (0, \infty) \to [0, 1]$ is continuous where $\mu, \upsilon, \theta \in \Omega$ and $t, s > 0$.

**Definition 1.3:** [5] Let $(\Omega, M, t)$ be fuzzy metric space then,
(1) A sequence $\{\mu_n\}_{n \in N}$ converges to $\mu \in \Omega$ if $\lim_{n\to\infty}(\mu_n, \mu, t) = 1$ for every $t > 1$.

(2) A sequence $\{\mu_n\}_{n \in N}$ in Ω is said to be Cauchy if, for each 0 < ϵ < 1, there is $n_0 \in N$ such that $M(\mu_n, \mu_m, t) > 1 - \epsilon$ for all n, $m \geq n_0$ where t > 0.
(3) If every Cauchy sequence is convergent, then (Ω, M, ∗) is complete.

**Definition 1.4:** [17] Let Ω be a nonempty set. Two mappings A, B: Ω → Ω are said to be weakly compatible if ABμ = BAμ for all μ ∈ Ω such that Aμ = Bμ.
**Definition 1.5:** [7] If the A, B self-map of a fuzzy metric space (Ω, M, ∗) are said to be occasionally weakly compatible if and only if there is a point μ ∈ Ω which is a coincidence point of A and B at which point A and B commute.
**Lemma 1.6:** [13] If μ, υ ∈ Ω and for a positive number k < 1, and M (μ, υ, kt) ≥ M (μ, υ, t) then μ = υ.
**Lemma 1.7:** [7] Let A, and B occasionally weakly compatible self-maps of Ω. If A and B have a unique point of coincidence, ω = Aμ = Bυ, then ω is the unique common fixed point of A and B.

## 2. Main Result

**Theorem 2.1:** Let A, B, C, D, E, F, G, and H be self-mapping of a complete fuzzy metric space (Ω, M, ∗) satisfying the following conditions:
(1) (GA, CD) and (HB, EF) are Occasionally Weakly Compatible.
(2) $A\mu = A^2\mu$, $B\mu = B^2\mu$ for all μ ∈ Ω and (G, A), (H, B), (E, F), (HB, F), and (D, GA) are commuting pairs.
(3) For ψ: [0, 1] → [0, 1] and there exists k ∈ (0, 1) such that

$$M(GA\mu, HB\upsilon, kt) \geq \psi\left\{\begin{array}{c} M(CD\mu, EF\upsilon, t), M(CD\mu, GA\mu, t), M(CD\mu, HB\upsilon, t), \\ \frac{M(CD\mu,\ GA\mu, t) + M(CD\mu, EF\upsilon, t)}{1 + M(GA\mu, EF\upsilon, t)}, M(GA\mu, EF\upsilon, t), \\ \frac{\alpha M(CD\mu, GA\mu, t) + \beta M(HB\upsilon, EF\upsilon, t) + \gamma M(GA\mu, EF\upsilon, t)}{\alpha + \beta + \gamma M(HB\upsilon, CD\mu, t)}, \\ \frac{\alpha M(GA\mu, HB\upsilon, t) + \beta M(HB\upsilon, CD\mu, t) + \gamma M(GA\mu, EF\upsilon, t)}{\alpha + \beta + \gamma} \end{array}\right\} \quad (1)$$

Where μ, υ ∈ Ω and α, β, γ ≥ 0 but α+β +γ ≠ 0. Then A, B, C, D, E, F, and G have a unique common fixed point.
**Proof:** (GA, CD) and (HB, EF) are Occasionally Weakly Compatible, let there exist points ζ, η ∈ Ω such that GAζ = CDζ and HBη = EF η. By inequality (1), we have

$$M(GA\zeta, HB\eta, kt) \geq \psi\left\{\begin{array}{c} M(GA\zeta, HB\eta, t), M(GA\zeta, GA\zeta, t), M(GA\zeta, HB\eta, t), \\ \frac{M(GA\zeta,\ GA\zeta, t) + M(GA\zeta, HB\eta, t)}{1 + M(GA\zeta, HB\eta, t)}, M(GA\zeta, HB\eta, t), \\ \frac{\alpha M(GA\zeta, GA\zeta, t) + \beta M(HB\eta, HB\eta, t) + \gamma M(GA\zeta, HB\eta, t)}{\alpha + \beta + \gamma M(HB\eta, GA\zeta, t)}, \\ \frac{\alpha M(GA\zeta, HB\eta, t) + \beta M(HB\eta, GA\zeta, t) + \gamma M(GA\zeta, HB\eta, t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$M(GA\zeta, HB\eta, kt) \geq \psi\left\{\begin{array}{c} M(GA\zeta, HB\eta, t), 1, M(GA\zeta, HB\eta, t), \\ M(GA\zeta, HB\eta, t), \\ \frac{\alpha + \beta + \gamma M(GA\zeta, HB\eta, t)}{\alpha + \beta + \gamma M(HB\eta, GA\zeta, t)}, \\ \frac{\alpha M(GA\zeta, HB\eta, t) + \beta M(HB\eta, GA\zeta, t) + \gamma M(GA\zeta, HB\eta, t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$M(GA\zeta, HB\eta, kt) \geq \psi\left\{\begin{array}{c} M(GA\zeta, HB\eta, t), 1, M(GA\zeta, HB\eta, t), \\ M(GA\zeta, HB\eta, t), \\ \frac{\alpha + \beta + \gamma M(GA\zeta, HB\eta, t)}{\alpha + \beta + \gamma M(HB\eta, GA\zeta, t)}, \\ \frac{(\alpha + \beta + \gamma) M(GA\zeta, HB\eta, t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$M(GA\zeta, HB\eta, kt) \geq \psi\{M(GA\zeta, HB\eta, t), 1, M(GA\zeta, HB\eta, t), M(GA\zeta, HB\eta, t), 1, M(GA\zeta, HB\eta, t)\}$$
$$\Rightarrow M(GA\zeta, HB\eta, kt) \geq M(GA\zeta, HB\eta, kt)$$

Therefore, GAζ = CDζ = HBη = EF η. Let Γ ∈ Ω such that GAΓ = CDΓ then by inequality (1) we have GAΓ = CDΓ = HBη = EF η. So GAζ = GAΓ and GAζ = EF ζ = ω are unique coincidences of GA and EF then by lemma 1.7, ω is a common fixed point of GA and EF. For Uniqueness let there be another fixed point ω∗ such that ω ≠ ω∗ then by inequality (1) we have

$$M(\omega, \omega^*, kt) = M(GA\omega, HB\omega^*, kt) \geq \psi\left\{\begin{array}{c} M(CD\omega, EF\omega^*, t), M(CD\omega, GA\omega, t), M(CD\omega, HB\omega^*, t), \\ \frac{M(CD\omega,\ GA\omega, t) + M(CD\omega, EF\omega^*, t)}{1 + M(GA\omega, EF\omega^*, t)}, M(GA\omega, EF\omega^*, t), \\ \frac{\alpha M(CD\omega, GA\omega, t) + \beta M(HB\omega^*, EF\omega^*, t) + \gamma M(GA\omega, EF\omega^*, t)}{\alpha + \beta + \gamma M(HB\omega^*, CD\omega, t)}, \\ \frac{\alpha M(GA\omega, HB\omega^*, t) + \beta M(HB\omega^*, CD\omega, t) + \gamma M(GA\omega, EF\omega^*, t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$\geq \psi \left\{ \begin{array}{c} M(\omega,\omega^*,t), M(\omega,\omega,t), M(\omega,\omega^*,t), \\ \frac{M(\omega,\omega,t)+M(\omega,\omega^*,t)}{1+M(\omega,\omega^*,t)}, M(\omega,\omega^*,t), \\ \frac{\alpha M(\omega,\omega,t)+\beta M(\omega^*,\omega^*,t)+\gamma M(\omega,\omega^*,t)}{\alpha+\beta+\gamma M(\omega^*,\omega,t)}, \\ \frac{\alpha M(\omega,\omega^*,t)+\beta M(\omega^*,\omega,t)+\gamma M(\omega,\omega^*,t)}{\alpha+\beta+\gamma} \end{array} \right\}$$

$$\geq \psi \left\{ \begin{array}{c} M(\omega,\omega^*,t), 1, M(\omega,\omega^*,t), \\ \frac{1+M(\omega,\omega^*,t)}{1+M(\omega,\omega^*,t)}, M(\omega,\omega^*,t), \\ \frac{\alpha+\beta+\gamma M(\omega,\omega^*,t)}{\alpha+\beta+\gamma M(\omega^*,\omega,t)}, \\ \frac{(\alpha+\beta+\gamma) M(\omega,\omega^*,t)}{\alpha+\beta+\gamma} \end{array} \right\}$$

$$\geq \psi\{M(\omega,\omega^*,t), 1, M(\omega,\omega^*,t), 1, M(\omega,\omega^*,t), 1, M(\omega,\omega^*,t)\} \Rightarrow M(\omega,\omega^*,kt) \geq \psi\, M(\omega,\omega^*,kt)$$

By the lemma 1.7, we get ω = ω∗. Hence, ω∗ is a common fixed point of GA, CD, HB, and EF. Since (G, A),(H, B),(E, F),(HB, F), and (D, GA) are commuting pairs, and $A\mu = A^2\mu$, $B\mu = B^2\mu$ for all μ ∈ Ω this implies that ω∗ is a common fixed point of A, B, C, D, E, F, G and H.

**Theorem 2.2:** Let A, B, C, D, E, F, G, and H be self-mapping of a complete fuzzy metric space (Ω, M, ∗) satisfying the following conditions:

(1) (GA, CD) and (HB, EF) are Occasionally Weakly Compatible.

(2) $A\mu = A^2\mu$, $B\mu = B^2\mu$ for all μ ∈ Ω and (G, A), (H, B), (E, F), (HB, F), and (D, GA) are commuting pairs.

(3) For ψ: [0, 1] → [0, 1] and there exists k ∈ (0, 1) such that

$$M(GA\mu, HB\upsilon, kt) \geq \psi\min \left\{ \begin{array}{c} M(CD\mu, EF\upsilon, t), M(CD\mu, GA\mu, t), M(CD\mu, HB\upsilon, t), \\ \frac{M(CD\mu, GA\mu, t)+M(CD\mu, EF\upsilon, t)}{1+M(GA\mu, EF\upsilon, t)}, M(GA\mu, EF\upsilon, t), \\ \frac{\alpha M(CD\mu, GA\mu, t)+\beta M(HB\upsilon, EF\upsilon, t)+\gamma M(GA\mu, EF\upsilon, t)}{\alpha+\beta+\gamma M(HB\upsilon, CD\mu, t)}, \\ \frac{\alpha M(GA\mu, HB\upsilon, t)+\beta M(HB\upsilon, CD\mu, t)+\gamma M(GA\mu, EF\upsilon, t)}{\alpha+\beta+\gamma} \end{array} \right\} \quad (2)$$

Where μ, ν ∈ Ω and α, β, γ ≥ 0 but α+β +γ ≠ 0. Then A, B, C, D, E, F, and G have a unique common fixed point.

**Proof:** (GA, CD) and (HB, EF) are Occasionally Weakly Compatible, let there exist points $\rho, \delta \in \Omega$ such that $GA\rho = CD\rho$ and $HB\delta = EF\delta$. By inequality (2), we have

$$M(GA\rho, HB\delta, kt) \geq \psi\min \left\{ \begin{array}{c} M(GA\rho, HB\delta, t), M(GA\rho, GA\rho, t), M(GA\rho, HB\delta, t), \\ \frac{M(GA\rho, GA\rho, t)+M(GA\rho, HB\delta, t)}{1+M(GA\rho, HB\delta, t)}, M(GA\rho, HB\delta, t), \\ \frac{\alpha M(GA\rho, GA\rho, t)+\beta M(HB\delta, HB\delta, t)+\gamma M(GA\rho, HB\delta, t)}{\alpha+\beta+\gamma M(HB\delta, GA\rho, t)}, \\ \frac{\alpha M(GA\rho, HB\delta, t)+\beta M(HB\delta, GA\rho, t)+\gamma M(GA\rho, HB\delta, t)}{\alpha+\beta+\gamma} \end{array} \right\}$$

$$M(GA\rho, HB\delta, kt) \geq \psi\min \left\{ \begin{array}{c} M(GA\rho, HB\delta, t), 1, M(GA\rho, HB\delta, t), \\ M(GA\rho, HB\delta, t), \\ \frac{\alpha+\beta+\gamma M(GA\rho, HB\delta, t)}{\alpha+\beta+\gamma M(HB\delta, GA\rho, t)}, \\ \frac{\alpha M(GA\rho, HB\delta, t)+\beta M(HB\delta, GA\rho, t)+\gamma M(GA\rho, HB\delta, t)}{\alpha+\beta+\gamma} \end{array} \right\}$$

$$M(GA\rho, HB\delta, kt) \geq \psi\min \left\{ \begin{array}{c} M(GA\rho, HB\delta, t), 1, M(GA\rho, HB\delta, t), \\ M(GA\rho, HB\delta, t), \\ \frac{\alpha+\beta+\gamma M(GA\rho, HB\delta, t)}{\alpha+\beta+\gamma M(HB\delta, GA\rho, t)}, \\ \frac{(\alpha+\beta+\gamma) M(GA\rho, HB\delta, t)}{\alpha+\beta+\gamma} \end{array} \right\}$$

$$M(GA\rho, HB\delta, kt) \geq \psi\min\{M(GA\rho, HB\delta, t), 1, M(GA\rho, HB\delta, t), M(GA\rho, HB\delta, t), 1, M(GA\rho, HB\delta, t)\}$$
$$\Rightarrow M(GA\rho, HB\delta, kt) \geq \psi M(GA\rho, HB\delta, kt)$$

Therefore, $GA\rho = CD\rho = HB\delta = EF\delta$. Let $\Gamma^* \in \Omega$ such that $GA\Gamma^* = CD\Gamma^*$ then by inequality (2) we have $GA\Gamma^* = CD\Gamma^* = HB\delta = EF\delta$. So $GA\rho = GA\Gamma^*$ and $GA\rho = EF\rho = \Theta$ are unique coincidences of GA and EF then by lemma 1.7, $\Theta$ is a common fixed point of GA and EF. For Uniqueness let there be another fixed point $\Theta^*$ such that $\Theta \neq \Theta^*$ then by inequality (2) we have

$$M(\theta,\theta^*,kt) = M(GA\theta, HB\theta^*, kt) \geq \psi\min\left\{\begin{array}{c} M(CD\theta, EF\theta^*, t), M(CD\theta, GA\theta, t), M(CD\theta, HB\theta^*, t), \\ \frac{M(CD\theta,\ GA\theta, t) + M(CD\theta, EF\theta^*, t)}{1 + M(GA\theta, EF\theta^*, t)}, M(GA\theta, EF\theta^*, t), \\ \frac{\alpha M(CD\theta, GA\theta, t) + \beta M(HB\theta^*, EF\theta^*, t) + \gamma M(GA\theta, EF\theta^*, t)}{\alpha + \beta + \gamma M(HB\theta^*, CD\theta, t)}, \\ \frac{\alpha M(GA\theta, HB\theta^*, t) + \beta M(HB\theta^*, CD\theta, t) + \gamma M(GA\theta, EF\theta^*, t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$\geq \psi\min\left\{\begin{array}{c} M(\theta,\theta^*,t), M(\theta,\theta,t), M(\theta,\theta^*,t), \\ \frac{M(\theta,\theta,t) + M(\theta,\theta^*,t)}{1 + M(\theta,\theta^*,t)}, M(\theta,\theta^*,t), \\ \frac{\alpha M(\theta,\theta,t) + \beta M(\theta^*,\theta^*,t) + \gamma M(\theta,\ \theta^*,t)}{\alpha + \beta + \gamma M(\theta^*,\theta,t)}, \\ \frac{\alpha M(\theta,\theta^*,t) + \beta M(\theta^*,\theta,t) + \gamma M(\theta,\theta^*,t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$\geq \psi\min\left\{\begin{array}{c} M(\theta,\theta^*,t), 1, M(\theta,\theta^*,t), \\ \frac{1 + M(\theta,\theta^*,t)}{1 + M(\theta,\theta^*,t)}, M(\theta,\theta^*,t), \\ \frac{\alpha + \beta + \gamma M(\theta,\ \theta^*,t)}{\alpha + \beta + \gamma M(\theta^*,\theta,t)}, \\ \frac{(\alpha + \beta + \gamma) M(\theta,\theta^*,t)}{\alpha + \beta + \gamma} \end{array}\right\}$$

$$M(\theta,\theta^*,kt) \geq \psi\min\{M(\theta,\theta^*,t), 1, M(\theta,\theta^*,t), 1, M(\theta,\theta^*,t), 1, M(\theta,\theta^*,t)\} \Rightarrow M(\theta,\theta^*,kt) \geq \psi\, M(\theta,\theta^*,kt)$$

By the lemma 1.7, we get $\theta = \theta^*$. Hence, $\theta^*$is a common fixed point of GA, CD, HB, and EF. Since (G, A),(H, B),(E, F),(HB, F), and (D, GA) are commuting pairs, and $A\mu = A^2\mu$, $B\mu = B^2\mu$ for all $\mu \in \Omega$ this implies that $\theta^*$is a common fixed point of A, B, C, D, E, F, G and H.

## 3. Conclusion

In this paper, we generalize the work of S. Chauhan and S. Kumar by applying eight self-mappings to an occasionally weakly compatible mapping of a complete fuzzy metric space in rational contraction.

## 4. References


1. Al-Thagafi MA, Shahzad N. Generalized I-nonexpansive selfmaps and invariant approximations. Acta Mathematica Sinica, English Series. 2008; 24:867-876.
2. Banach S. Sur les op´erations dans les ensembles abstraits et leur application aux ´equations int´egrales. Fundamenta mathematicae. 1922; 3(1):133-181.
3. Chauhan S, Kumar S. Fixed points of occasionally weakly compatible mappings in fuzzy metric spaces. Scientia Magna 2011; 7(2):22-31.
4. Chourasiya S, Shrivastava K. Common Fixed Point Theorem in Controlled Metric Spaces. Int. j. adv. multidisc. res. stud. 2024; 4(5):317-323.
5. George A, Veeramani P. On some results in fuzzy metric spaces. Fuzzy sets and systems. 1994; 64(3):395-399.
6. Gregori V, Sapena A. On fixed-point theorems in fuzzy metric spaces, Fuzzy sets and systems. 2002; 125(2):245-252.
7. Jungck G, Rhoades B. Fixed point theorems for occasionally weakly compatible mappings. Fixed point theory. 2006; 7(2):287-296.
8. Kaleva O, Seikkala S. On fuzzy metric spaces. Fuzzy sets and systems. 1984; 12(3):215-229.
9. Kanwal S, I¸sık H, Waheed S. Generalized fixed points for fuzzy and nonfuzzy mappings in strong b-metric spaces. Journal of Inequalities and Applications. 2024; 2024(1):22.
10. Karlsson A. A metric fixed point theorem and some of its applications. Geometric and Functional Analysis. 2024; 34(2):486-511.
11. Kramosil I, Mich´alek J. Fuzzy metrics and statistical metric spaces. Kybernetika. 1975; 11(5):336-344.
12. Mi˜nana JJ, Sostak A, Valero O. On metrization of fuzzy metrics and application to fixed point theory. Fuzzy Sets and Systems. 2023; 468:108625.
13. Mishra SN, Sharma N, Singh SL. Common fixed points of maps on fuzzy metric spaces. International Journal of Mathematics and Mathematical Sciences. 1994; 17(2):253-258.
14. Okeke GA, Francis D. Fixed point theorems for metric tower mappings in complete metric spaces. The Journal of Analysis. 2024; 32(2):949-991.
15. Schweizer B, Sklar A. Statistical metric spaces. Pacific J. Math. 1960; 10(1):313-334.
16. Shukla S, Dubey N, Mi˜nana JJ. Vector-Valued Fuzzy Metric Spaces and Fixed Point Theorems. Axioms. 2024; 13(4):252.
17. Sintunavarat W, Kumam P. Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. Journal of Applied mathematics. 2011; (1):637958.

18. Sonam Bhardwaj R, Mal J, Konar P, Sumalai P. Fixed point results in soft probabilistic metric spaces. The Journal of Analysis, 2024, 1-28.
19. Subrahmanyam PV. A common fixed point theorem in fuzzy metric spaces. Information Sciences. 1995; 83(3-4):109-112.
20. Zadeh LA. Fuzzy sets, Information and Control, 1965.