Int. j. adv. multidisc. res. stud. 2025; 5(3):422-426

Received: 26-03-2025
Accepted: 06-05-2025

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

A Financial Derivatives of Asset and Liability: An Improved Black Scholes Model

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DOI: https://doi.org/10.62225/2583049X.2025.5.3.4254
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Abstract
This work provides an improved Black-Scholes model by incorporating inflation rate, dividend yield and risk free rate. The resulting pricing equation for derivatives and, in particular, the formula for European call options is then shown to depend explicitly on the drift of the underlying asset, which is following a geometric Brownian motion. It is visualized that with the present model, the predicted results by the model could be closer to real data. The adjusted pricing model could partly also explain the mystery of volatility smile. The present model also provides answers to many finance professionals and academics who have been intrigued by the effect of inflation rate, dividend yield and risk free of their asset. The model provides generally different fair values for financial derivatives compared to the Black-Scholes model. In particular, the present model predicts that the original Black-Scholes model tends to undervalue for example European call options.

Keywords: Options Pricing, Financial Derivatives, Efficient Market Hypothesis, Martingale, Feynman-Kac, Black-Scholes

1. Introduction

The pricing of financial derivatives underwent a significant transformation in 1973 with the introduction of the groundbreaking Black-Scholes framework (Black & Scholes, 1973) [5]. This seminal model provides a linear parabolic partial differential equation (PDE) that governs the value of financial instruments, such as European call options. Notably, an explicit formula is available to calculate the option's value, contingent upon key parameters. To derive these formulas, the Black-Scholes PDE is typically transformed into a constant coefficient PDE, leveraging techniques like Fourier analysis. The essential parameters required for this calculation include: Risk-free interest rate, Volatility and Exercise price (strike price). The Black-Scholes model's performance has been scrutinized in empirical terms, as it often fails to accurately predict true market values of options. Despite its limitations, the model remains a widely used benchmark for traders, financial markets professionals, and risk managers. However, several studies, including those by Bergman (1982) [4] and Musiela and Rutkowski (2005) [16], have challenged the model's underlying assumptions. Specifically, they argue that the hedging portfolio in the original model is not self-financing. A more fundamental critique is that the model's premise is overly narrow, rendering the delta-hedging argument largely irrelevant to practitioners. One of the most counterintuitive aspects of the Black-Scholes model is the independence of the option's fair price from the underlying asset's drift. This result stems from the model's framework, which assumes a geometric Brownian motion with non-zero drift for the underlying asset. Paradoxically, the model suggests that the value of a call option is unaffected by the underlying asset's drift, contrary to the intuitive expectation that a higher positive drift would increase the option's value. The Black-Scholes model operates under the assumption of a risk-neutral market, where the asset's drift is effectively equivalent to the risk-free rate. The mathematical justification for the irrelevance of the true drift lies in the hedging portfolio approach. Specifically, the model involves shorting the underlying asset by an amount equal to the option's delta. As a result, any increase in the asset's value due to a higher positive drift is offset by a corresponding decrease in the short position's value, rendering the hedging portfolio instantaneously risk-free and yielding the risk-free rate. The option's value is then determined as a residual to maintain the portfolio's local risk-free status. However, this approach has two significant limitations. Firstly, the Black-Scholes model's fair price for a call option is only relevant to the hedging portfolio holder, not to other market participants. For instance, a speculator holding a long call option on a blue-chip company's stock is unlikely to hold a short position in the underlying asset that exactly offsets the option's delta. Consequently, the model's assumptions do not reflect the reality of most market participants' portfolios, limiting its applicability. Beyond the conceptual issues, the Black-Scholes model faces a well-known technical challenge related to self-financing portfolios. The model assumes that the hedging portfolio is self-financing, meaning that its value changes solely due to fluctuations in the option and underlying asset prices, without any external funding flows. However, this assumption has been disputed by researchers such as Bergman (1982) [4] and Bartels (1995) [3], who argue that it may not hold in practice. If the hedging portfolio is not truly self-financing, the rate of return may not be riskless, which would undermine the model's fundamental assumptions. While this flaw does not necessarily invalidate the Black-Scholes model entirely, as noted by Bana (2007) [2], it does highlight significant concerns. Additionally, the model's assumption of geometric Brownian motion for the underlying asset may not perfectly capture real-world dynamics, although this issue is secondary to the self-financing problem. Research on imperfect hedging portfolios due to market incompleteness has shown that contingent claim values lie within certain hedging bounds (Hao, 2008) [12]. Other studies have explored the impact of costly short-selling on bid-ask spreads (Atmaz & Basak, 2019) and developed equal risk pricing rules for incomplete markets (Guo & Zhu, 2017 [11]; Marzban *et al.*, 2022 [15]). However, these approaches do not address the fundamental issue of delta-hedging, where holding a long call option is unlikely to be perfectly offset by shorting the underlying asset.

2. Properly Anticipated Prices in the Options Pricing Framework: A Generalized Framework for Derivative Pricing this model proposes a novel approach to options pricing, building on the efficient markets hypothesis (EMH) as introduced by Paul Samuelson (1965 [19], 1973 [20]). We argue that the traditional Black-Scholes framework, which relies on hedging portfolios, should be generalized to better align with the principles of efficient markets. the traditional delta-hedging approach has limitations, and a more suitable framework involves determining the appropriate discount rate for an efficient financial market. Notably, an asset price following geometric Brownian motion with drift is not a martingale in its raw form. However, when properly discounted, the price process exhibits martingale properties, aligning with the principles outlined by Samuelson (1973) [20] and his property is a direct consequence of the efficient market hypothesis, as established by Fama (1965 [9], 1970 [10]). A fundamental assumption follows: in an efficient market, a properly discounted asset price process should exhibit martingale behavior. Consider an asset price A that evolves according to geometric Brownian motion:

$$\frac{dA}{dt} = (r+\delta+\pi)A - \sigma A\left(\frac{dW}{dt}\right) \quad 0 \le t \le T \tag{1}$$

$r > 0$ is the risk-free rate, $\delta > 0$ is the dividend yield, $\pi > 0$ is the inflation rate $\sigma > 0$ here, σ represents the volatility, and W denotes a standard Brownian motion. We focus on valuing a generic financial derivative based on this asset. Consistent with Samuelson's (1973) [20] formulation of the efficient market hypothesis, as reviewed by LeRoy (1989) [13], we impose the martingale condition on the discounted price process. Given that for a geometric Brownian motion, the expectation for the price at future time T > 0 is given by:

$$Et(A_T) = A_t e\ (T\ t), \tag{2}$$

Where Et is an expectation operator with respect to the probability measure generated by the Brownian motion. market in order to obtain:

$$A_T = A_0 e^{(r+\delta+\pi)T} \tag{3}$$

This market efficiency requirement implies that the expected discounted future price of an asset equals its current price, reflecting the market's collective risk aversion and incorporating all relevant information. For a financial derivative with a terminal payoff at time T and payoff function $g(A_T)$, we seek a fair value that is independent of individual market participants' risk preferences. Instead, we discount the derivative's payoff using the market discount function and evaluate the expectation under the physical probability measure;

$$P(A(t),t) = E_t e^{-(r+\delta+\pi)(T-t)}, \tag{4}$$

where P(x(t),t) is the fair value of the financial derivative at time t when the asset has some known price x(t). It is straightforward to use the Feynman-Kac formula (Pavliotis 2014) [18] to write down the partial differential equation describing the evolution of the value of the financial derivative:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 P}{\partial A^2} + (r+\delta+\pi)A\frac{\partial P}{\partial s} - (r+\delta+\pi)P = A \tag{5}$$

With the initial condition $P(A,T) = (A_T)$. Notice that the only difference to the canonical Black-Scholes partial differential equation is that the risk-free rate is replaced with the discount rate reflecting the efficiency of the financial market. As an instructive example, consider now pricing a plain vanilla European call option. The payoff is $(A_T) = \max(A_T\ K, 0)$, where K > 0 is the exercise or strike price of the call option maturing at time T > t. As everything else is the same as in the Black-Scholes pricing PDE, we can deduce the price for a European call option easily by using the well-known formulas and by just replacing the risk-free rate in the formulas with the drift of the underlying asset: $P(Xt,t) = P(A,t) = Ke^{-(r+\delta+\pi)(T-t)\emptyset(-d_2)} - A\emptyset(-d_1)$

Where N is the cumulative distribution function of the standard normal variable, and

$$d_1 = \frac{\ln(\frac{A}{K}) + (r+\delta+\pi) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(\frac{A}{K}) + (r+\delta+\pi) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$= d_1 - A\sqrt{T-t}$$

The resulting pricing formula bears similarity to the one presented by James Boness (1964) [6]. Notably, the European call option's price now explicitly depends on the asset's drift, aligning with intuitive expectations. The option's rho is positive, indicating that an increase in drift enhances the value of a long call. The primary distinction from the canonical Black-Scholes model lies in replacing the risk-free rate with the underlying asset's drift. This adjustment, which affects the strike price, may help mitigate the volatility smile phenomenon, as discussed by Derman and Miller (2016) [8].

3. Result and Discussions

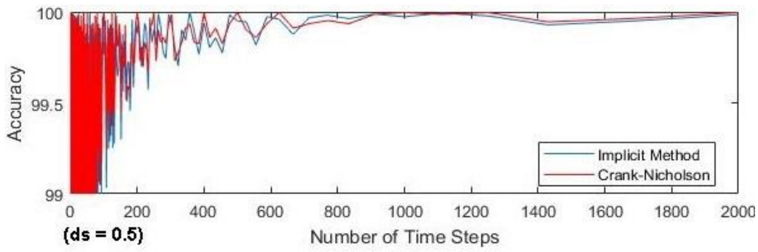
We propose that in an efficient market, the risk-adjusted discounted expected price of an asset following geometric Brownian motion equals its current price. This assumption implies that the correct risk-adjusted discount rate is the asset's drift. Consequently, derivatives pricing can be formulated as a conditional expectation discounted at this rate, reflecting market efficiency. To numerically solve the resulting partial differential equation, we employ the Finite Difference Method, discretizing the asset's price and time domains into a mesh. The mesh's vertical axis represents the asset price, while the horizontal axis represents time. Each point in this grid will have a horizontal index j and a vertical index i. At any given point, $j\partial A$ is equals to the asset price and $j\partial t$ is equals to the time. There exist three boundary conditions for the mesh, and it is with these that all calculation will be made.

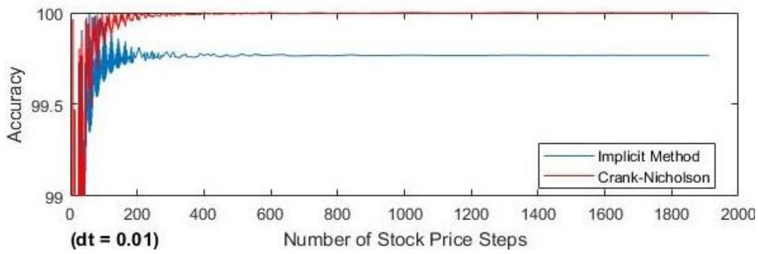
First of all, there exist an upper boundary where the option values for maximal Asset value in the mesh Amax will be calculated. Amax will be chosen large enough so that the put option is equal to zero according to the payoff function $max(K - s, 0)$. For this boundary condition, the option value will therefore be equals to zero for all times in the mesh. For these lower boundary conditions, the asset price will be equals to zero $(A = 0)$. Consequently, the option price will be equals to K for this boundary condition according to the payoff function. Therefore, for this boundary condition, the option value will be equal to the discounted value of the strike price for all times in the mesh. The boundary condition, which is the option price at expiration date, will be calculated using the payoff function. The time is known at expiration, $t = T$, and the option can be calculated for all the assets prices for this boundary condition. To obtain the price at $t = 0$, the option prices will be calculated for each time step backward from $t = T$.

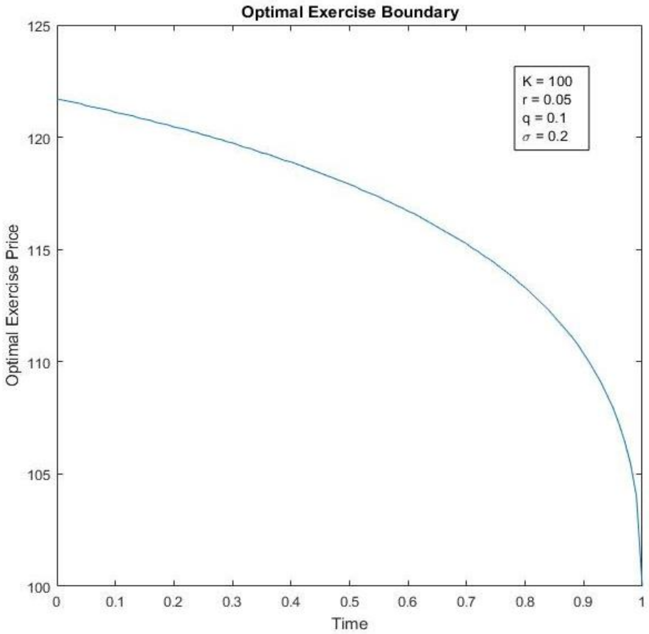
Applying the Finite Difference approximation on the derivatives in the Black Scholes partial Differential Equation on the price time mesh gives the following approximation of the derivatives.

Table 1: A comparison between the performance of the explicit method, implicit method and the Crank-Nicholson method for an European option with $K = 100$, $r = 0.05$, $\sigma = 0.2$ $\delta = 0.05, \pi = 0.02$ and $T = 1$. The computational time is the average computational time for 100 trials. Explicit: dt = 0.0001 and ds = 1. Implicit: dt = 0.0001 and ds = 0.05. Crank-Nicholson: dt = 0.001 and ds = 0.05.

		Explicit Method			**Implicit Method**			**Crank-Nicholson**		
A	**True Value**	Value	Accuracy	Time	Value	Accuracy	Time	Value	Accuracy	Time
90	2.7594	2.7591	99.99%	0.0341	2.7599	99.97%	0.3063	2.7595	100.00%	0.0402
95	4.3136	4.3128	99.98%	0.3242	4.3135	100.00%	0.3093	4.3136	100.00%	0.0396
100	6.1912	6.1900	99.98%	0.0343	6.1906	99.99%	0.3108	6.1913	100.00%	0.3287
105	8.6109	8.6094	99.98%	0.0241	8.6098	99.98%	0.2091	8.6112	100.00%	0.3266
110	11.5506	11.5488	99.98%	0.0241	11.5491	99.99%	0.1950	11.515	99.99%	0.0347
115	14.9579	14.9562	99.99%	0.0256	14.9564	99.99%	0.2123	14.9590	99.99%	0.0399
120	18.7630	18.7616	99.99%	0.0265	18.7616	99.99%	0.2295	18.7642	99.99%	0.0441
125	22.8905	22.8892	99.99%	0.0277	22.8894	100.00%	0.2418	22.8919	99.99%	0.0455
130	27.2690	27.2681	100.00%	0.0296	27.2682	100.00%	0.2816	27.2705	99.99%	0.0502







In figure 2 it can be found the optimal exercise price decreases over time. This is explained, as mentioned, by the fact that the time value decrease as the option approaches the expiration date. Furthermore, the optimal exercise price is equal to the strike price at maturity, since the time value will be equal to zero at $t = T$. Our findings indicate that the hedging portfolio approach is overly restrictive. The Black-Scholes model's fair price for an option relies on a hedging portfolio that yields the risk-free rate, forcing the derivative's value to conform to this constraint. However, this delta-hedging approach has limitations, as it assumes option holders maintain a perfectly hedged portfolio. For investors without such a portfolio, changes in the underlying asset's drift can impact the option's value. Our analysis, grounded in efficient market theory, demonstrates mathematically why drift should influence derivative pricing.

4. Conclusion

The proposed model predicts that the price of a European call option depends explicitly on the underlying asset's drift. This approach may yield more accurate empirical results when comparing theoretical prices to actual market prices. Additionally, it offers a new perspective on the volatility smile phenomenon, warranting further examination. The higher discount rate in our model, compared to the original Black-Scholes framework, implies that the fair value of a financial derivative generally depends on the underlying asset's drift. For a European call option holder, this means that options on assets with higher instantaneous returns or drifts are more valuable. This result stems from our approach, which values options based on market risk aversion rather than a hedging portfolio. The well-documented discrepancy between option market data and Black-Scholes predictions suggests a need for alternative models. Our model may provide more accurate fair values for financial derivatives, potentially benefiting investors. Furthermore, if our approach outperforms the original Black-Scholes model, implied volatilities would likely change. Notably, our model incorporates the underlying asset's drift, which can be estimated from data, similar to the risk-free interest rate in the original model. The present model suggests that when pricing financial derivatives with an underlying following geometric Brownian motion, one should use the pricing PDE (5) instead of the original Black-Scholes PDE. In similar fashion, for European call options, one should use pricing Formulas (6) – (8). Finally, it is indeed interesting to note that the pricing formula in the present approach is similar to the pricing formula of (Boness 1964) [6], see also the discussion in (O'Brien and Selby 1986) [17].

5. Acknowledgement

The authors acknowledge the exceptional research facilities at Ladoke Akintola University of Technology's Department of Pure and Applied Mathematics.

6. Data Availability Statement

Not applicable

7. Conflicts of Interest

The author declares no conflict of interest

8. Ethical statement

This work is exceptional and has not been published in full or in part anywhere else.

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