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Solution of Fourth Order Wave Equation and Applications to Artificial Intelligence

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Abstract
In this paper, we provide a solution to a fourth order Ordinary Differential Equation (ODE) in a transmission system. The main focus in this work is to examine the wave equation arising due to collapse of shafts in power transmission systems and analyze it using Lie symmetry techniques.

Keywords: Lie Symmetry, Wave Equation, Power Transmission, ODE

Introduction

Differential equations are prevalent in various natural phenomena and technological challenges and their solutions have been very instrumental in the emerging area of Artificial Intelligence (AI). These equations pertain to the behavior of specific unidentified dependent variables at a particular point, such as time or place [9]. These mathematical differential equations, which have been defined, possess solvable derivatives. Multiple endeavors have been undertaken to resolve these differential equations by employing numerical methods, namely utilizing the finite difference approach. Nevertheless, numerical solutions are merely estimations that rely on specific initial boundary conditions. Consequently, they are not appropriate for resolving problems related to mechanical vibrations that require precise values [2]. Therefore, this study may offer an alternate analytical approach for solving equations related to the failure of shafts in power transmission systems and other mechanical issues, utilizing Lie symmetry on Lie Group Transformation (LGT). This has the potential to enhance understanding in the field of applied mathematics and serve as a pathway for additional research. Moreover, this study could assist engineers in comprehending the oscillation of a vehicle caused by road fissures [8]. This could facilitate engineers in enhancing the design of automobiles and other mobile engines. The conventional integration methods for ODEs, according to Norwegian mathematician Lie, derive their solutions using the symmetries of the equations [7]. Therefore, it is possible to make any differential equation reveal its symmetries, which are then utilized to build the exact solutions. Consequently, he came to the realization that both techniques might be combined and expanded upon to create a generic integration process predicated on the differential equations invariance under a continuous set of symmetries [4]. To study nonlinear differential equations and produce their exact and implicit solutions in a fully algorithmic manner, symmetry analysis of differential equations was created. Lie found that studying the associated vector fields, or infinitesimal generators, was the most effective method for comprehending Lie groups because an infinitesimal generator contains all of the information about a LGTs. Figuring out the small changes from the symmetry group gives us the group generator, which is the same as creating the group [6]. The Lie's algorithm used to analyze the symmetry of differential equations was further advanced through the efforts of [11] in the late 1950's. Galois' hypothesis, which became known in 1850, provided Lie with the most significant guidance for his work [1]. Following his discovery that the symmetry of an ODE is a basic idea that enables solving it, Lie went on determining its determining equation [18]. Lie's group theory is now used in many kinds of equations, both simple and complex, to help model various physical or abstract situations. In the mid-1980s, Lie group theory became popular because of the work done by Leach, a South African with his student. After that, many researchers from different fields, where differential equations are important, like mechanical vibrations, ecology. Due to its wide nature of applicability, nowadays Lie group analysis has attracted many researchers in various fields. Therefore, it can be claimed that symmetry approaches connect to a number of ODE-related subjects, such as the usage of Laplace transformation and methods for varying parameters and indeterminate coefficients [6]. Canonical variable is a strategy that relies on point symmetries to build transformations that simplify the equation before solving it. The first integrals and first order partial differential equation methods are strongly associated with the canonical variable's method. Canonical transformations can be computed given a symmetry group. enabling the integration of first-order ODEs or lowering the order of ODEs of higher order. Furthermore, if a two-dimensional symmetry group is admitted by a second-order equation, we can directly modify the variables such that the equation becomes integrable rather than successively reducing the order. This modification of variables was dubbed the canonical variables approach by [7]. A collection of variables is designated as canonical by Lie if the equation can be simplified and allows for integration by quadrature. Let's look at a set of transformations that comprise two generators, V_1 and V_2, in more detail. If the following relations do not exist: $V_1 = cV_2$, where c is a constant; and $(V_1, V_2) = c_1V_1 + c_2V_2$, where c_1 and c_2 are also constants, then two infinitesimal transformations, V_1 and V_2, are independent of one another. Assuming that the product may be represented differently by each of the two independent transformations, the second relation can be made simpler. Very few equations admit enough point symmetries to allow for reduction to quadrature in an effort to get around this restriction [10]. One such extension results from the observation of what are known as hidden symmetries, which are point symmetries that unintentionally develop. It has been demonstrated that solutions of certain equations lacking sufficient Lie point symmetries with the corresponding Lie algebras can be reached through hidden symmetries. Type I hidden symmetries can arise when an equation's order is increased, while Type II hidden symmetries can arise when an equation's order is decreased [2]. We provide a comprehensive summary of the research conducted by different academics about the derivation of equations for fourth-order differential equations in dynamic motion. Our focus is mostly be on the elucidation of the general equation and its correlation with the failure of shafts in power transmission systems.

Literature review

Many studies have considered a systematic program of applying the Lie continuous group of transformations methods of up to the third order [1]. In this work we have looked up to the fourth order prolongation. We have applied the fourth extension to expand the wave equation. The work of [2] on prolongations, studied an ODE with a scalar coefficient which had a given number of Lie symmetries contained in a total of seven equivalent classes. However, in our work we look at a fourth order ODE with a fourth degree which admits a second order symmetry. According to [10], Lie developed the idea of continuous groups of transformations which has been termed as Lie groups; named after him to consolidate and extend a number of specialized approaches to solving ODEs. Lie's work has systematically connected numerous topics and methods in ordinary differential equations. We now focus on the symmetry group**,** which is one of the most significant groups in relation to DEs. In [4] the study introduced the idea of generalized conditional symmetry, and further expanded on this approach. Group theory is used in all of these techniques for determining symmetries and related similarity reduction of a given PDE. The innovative characteristics of [5] direct algorithmic approach for identifying similarity reduction of PDEs are completely self-explanatory without the need for group analysis. A geometrical object's symmetry is defined by [6] as a transformation whose action appears to leave the object untouched. He also used symmetries to classify objects. Moreover, the work of [7] extended the idea of LGTs to non-local symmetries of DEs. They applied a systematic approach to identify a particular class of non-local symmetries, known as potential symmetries. Contributors to this discussion on rotating shafts as seen in [8] pointed out that the mapping techniques established significantly increased roles to DEs. Many scholars have studied the equations of shaft collapse as explained in different articles. For instance [9], provided a stability approach for the precise solution of a DE using symmetry group. The Burger's equation was solved by [11]. The author was able to identify every Lie group that Burger's equation admitted, and he then employed the symmetry transformation to identify every global solution that went along with every Lie group that the equation admitted [12]. In their work on prolongations, the authors used seven equivalency classes, each of which had three point Lie symmetries, as a base for their scalar second order ODEs. However, in our study, we examine a fourth-degree polynomial in y_3, a fourth order ODE that admits second order symmetry. The solution of a third order first degree non-linear ODE that arises was studied [14] in the study of Lie symmetry. In contrast, we examined a fourth order ODE in our study that is a fourth-degree polynomial in y_3 and admits second order symmetry. In a different study [15] examined novel single generators of Lie point symmetries that allow an ODE's order to be reduced once. He used two-point symmetries to describe a double reduction of order. In [16], the authors discovered first integrals for higher order ODEs and demonstrated how to lower an ODE's order by utilizing higher order symmetries. Additionally, he contrasted several ODE integration and supplementary techniques. The author handled dimensional analysis in great detail. In order to identify solutions of differential equations, different aspects of symmetries were described. The author focused on developing solutions and the first integral that resulted from these symmetries and integrating factors by using an explicit approach. The author included a full discussion of dimensional analysis in his work and used examples from physical and engineering difficulties, such as those involving heat conditions and wave propagation [17]. The author presents the reader with the Backingham pi-theorem, which presents the idea of invariance. The author was able to show how this results in generalizations by showing how boundary value problems are invariant under changing scaling. This gets the reader ready to think about differential equations' more general invariance under transformation groups [18]. The author demonstrated the process of locating higher order, acknowledged point, and contact symmetries. Additionally, he demonstrated how to expand the reduction process to include symmetries and use corresponding to identify admitted first integrals. utilizing initial integrals to get order reductions and integrating factors. This greatly broadens and stream- lines the traditional theorems for determining conservation laws, to include any ODE not just those that admit a variable principle [1]. Specifically, he demonstrated how to compute integrating factors using a variety of computational techniques that are similar to those used to compute variable that changes [8]. He made a clear comparison between the unique ways that admitted local symmetries and acknowledged integrating factors reduce order. He gave an ex- ample of how to solve boundary value problems using invariance under point symmetries. By examining their topological characteristics, he demonstrated that invariant solutions comprise separatrices and singular envelop solutions. He also developed an approach to create exceptional solutions, or invariant solutions, that arise from accepted techniques.

Research methodology

We mainly focus on methods for finding solutions that stay the same under certain group changes, called Lie groups. The space that contains the system's independent and dependent variables is influenced by the Lie groups. Building generators for infinitesimal transformations, prolongations, determining equations and integrating factors.

Results and discussion

In this section, we give the results of our study in the subsections below:

Invariant transformations

When all of the group's transformations result in a point on curve C mapping into another point on the curve, that curve is said to be invariant. This means that the solutions to a certain differential equation stay the same when certain transformations are applied, specifically under a smaller group of transformations that the system allows [5]. In order for the infinitesimal coefficients $\xi(x, y)$ and $\eta(x, y)$ to simultaneously disappear, the curve must form an orbit [9]. The equation below can be used to parametrically express a family of curves with one parameter $\phi(x, y) = C$, where the function $\phi(x, y)$ defines the family and C defines the parameter that labels various curves of the family [10]. We therefore say the family is invariant if the incase of each curve of it is another curve of the family [11]. Any specific value of λ has to be true for the image points (x^j, y^j). The solutions to these ODE systems are part of a smaller group within the larger group that the system allows. By decreasing points of in the set of DEs, these solutions are found. The most general solution for ψ in the aforementioned equations can be found by equating an arbitrary function of the two independent integrals to zero. When the first of these two equations is integrated, the result is $x^2 + y^2 = a^2$, where a^2 is the integration constant.

Invariance of differential equations

Known as invariant solutions of differential equations (DEs), invariant curves from the LGTs that these equations allow provide a helpful way to find their solutions. If there isn't a constant solution, we can make the difficulty of a regular differential equation easier by using the constants [9]. In general, for an n^{th} order ODE that is part of a r-parameter Lie group with a solvable Lie Algebra, its invariance can be explained by considering the following higher order ODE, $F_n(x, y, y_1, \ldots y_n)=0$. This equation can be transformed and solved using many techniques which are not numerical in nature.

Integrating factors and adjoint symmetry

The provided function takes the systems and multiplies them with the ordi- nary differential equation (ODE) to transform it into a precise form known as a total derivative form. Integrating factors provide a methodical way to reduce the complexity of an ODE by locating a primary integral [3]. Compared to reducing using point symmetries, the simpler ordinary differential equation (ODE) includes both the independent variables we get from it. Also, any starting integral can be found using an integrating part. This factor is decided by a math formula that looks at the variables and their rates of change up to a certain level. The factor is then multiplied using a regular differential equation (ODE) to change it into a specific form called a total derivative as shown in [10]. For instance, for an AS to be an IF, there must be sufficient and necessary additional determining equations. Consequently, the study of the first integral of ODEs heavily relies on AS [12]. Infinitesimal generators (IG) are properties of transformational Lie groups. Lie provided a technique for doing this. Suppose that a one parameter ε LGTs is parameterized in such a way that $\phi(a, b) = a + b$ gives its law of compositions, and $\varepsilon^{-1} = -\varepsilon$ and $[(\varepsilon) \equiv 1$. in terms of its infinitesimals ξ. Therefore, the IG of the one-parameter LGT.

Applications to artificial intelligence

The study of ODEs has been very useful in the emerging recent technologies. One of the considered cases is the AI sector. This is an area which has seen tremendous growth and attention of late. Solutions of ODEs have been applied in Ai to increase the speed and efficiencies of the applications being developed. The results of this study are very important particularly in robotics in determining the precision of the robots being developed and used to carry out surgeries in the medical field.

Conclusion

In this section, we conclude and give recommendations of the study. Using prolongation, we obtained the general solution of equation arising due to the collapse of shafts by use of integrating factors adjoint symmetries, invariant transformation of the problem and infinitesimal generators. We have developed characteristic equations, utilized invariance of equation differentials to lower order linear ODE which we have used to find the general solution. For future research, we recommend that obtaining the determining equations using extended generators is tedious, therefore an algorithm should be developed to solve the problem numerically. Also, further research should consider solving higher order mechanical vibration equations emanating from a heavy particle suspended from a coiled spring and oscillates in the vertical direction analytically or numerically.

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