Int. j. adv. multidisc. res. stud. 2025; 5(1):1241-1250

Received: 06-01-2025
Accepted: 16-02-2025

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

On Borel Map for Compact Sets in Plane of Entire Functions

[1] **Khaled AbdElkhalek,** [2] **Shawgy Hussein**
[1, 2] Department of Mathematics, College of Science, Sudan University of Science and Technology, Sudan

DOI: https://doi.org/10.62225/2583049X.2025.5.1.3790
Corresponding Author: **Khaled AbdElkhalek**

Abstract
We follow the well build method of a discussion shown by Paulo D. Cordaro, Giuseppe Della Sala and Bernhard Lamel [12] on the Borel map at a point $p^2 \in K$, where K is the polynomially convex hull of the compact set $K \subset C$, associating to every sequence of functions $f_r \in A^\infty(K)$ its formal Taylor series be at p^2. We show that it satisfies an alternative condition (provided that K fulfills a mild metric condition that): It is either surjective or injective, and the latter is the case if and only if $p^2 \in K$. We also discuss variants of this results for approximation by rational functions and a smooth version of the Mergelyan theorem.

Keywords: Borel Map, Mergelyan Theorem, Surjectivity

1. Introduction

There has been a growing interest in the Borel map associated to solutions to systems of (integrable) $PDEs$; (see [1, 2, 4, 5, 6, 7]) for an introduction to this notion and for some of the fascinating results relating (failure of) uniqueness properties of solutions to their richness.

The key raised from that study is the importance of the Fréchet algebra $A^\infty(K)$ for compact subsets $K \subset \mathbb{C}^m$. These algebras are the closures of the restrictions of entire functions to K, where we require uniform convergence of every derivative on K. So the polynomial hull of K is defined as the set

$$\hat{K} = \left\{ z^i \in \mathbb{C}^m : |P(z^i)| \le \sup_{q^2 \in K} |P(q^2)| \text{ for all } P \in \mathbb{C}\left[z_1^i, \dots, z_m^i\right] \right\}.$$

We define a "Borel map" $\mathfrak{b}_{K,p^2}$ for any point $p^2 \in \hat{K}$ by associating to a function its (formal) Taylor series at :

$$A^\infty(K) \ni f_r \mapsto \mathfrak{b}_{K,p^2} f_r = \sum_{\alpha \in \mathbb{Z}_+^m} \sum_r \sum_i \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f_r}{\partial z^{i\alpha}}(p^2)(z^i - p^2)^\alpha \in \mathbb{C}[[z^i - p^2]].$$

For every knowledge in the literature, we were not able to find a resolution to our main topic of interest even in the one variable case, namely, whether failure of surjectivity of the Borel map is equivalent to the existence of nontrivial complex structure of $\hat{K}$ at p^2-that is to say, the existence of a germ of positive-dimensional complex analytic set $X \subset \hat{K}$ such that $p^2 \in X$. We therefore need to show this question in the case $m = 1$. The purpose now is to give a rather complete solution under a rather mild metric condition on K:

Definition 1. We say that a compact set $K \subset \mathbb{C}$ is path-bounded if there exists $\epsilon \ge 0$ such that for any neighborhood U of K and any points $p^2, q^2 \in K$ there exists a rectifiable curve $\gamma^r \subset U$ of length at most $1 + \epsilon$ joining p^2 and q^2.

We can now state the main theorem (see [12]).

Theorem 1. Let $K \subset \mathbb{C}$ be a path-bounded compact set and $p^2 \in \hat{K}$. Then one and only one of the following properties hold:

(i) $\mathfrak{b}_{K,p^2}$ is surjective;

(ii) $\mathfrak{b}_{K,p^2}$ is injective and $\hat{K}$ is a neighborhood of p^2.

The alternative in this theorem is another strong indication that unique continuation (by which we mean the determination of a function by its formal expansion at a point) is a prerogative of holomorphic functions and that failure of unique continuation already leads to surjectivity of the Borel map.
We also develop a parallel theory for the algebra $M^{\infty}(K)$ of functions which can be approximated uniformly on K in every derivative by rational functions with poles outside of K. Now we can define the corresponding Borel map $\mathfrak{b}^{\ddagger}_{K,p^2}$ for every $p^2 \in K$; and the corresponding alternative is the following (see [12]):

Theorem 2: Let $K \subset \mathbb{C}$ be a path-bounded compact set and assume that $\mathbb{C} \setminus K$ has only finitely many components and $p^2 \in K$. Then one and only one of the following properties holds:
(i) $\mathfrak{b}^{\ddagger}_{K,p^2}$ is surjective;
(ii) $\mathfrak{b}^{\ddagger}_{K,p^2}$ is injective and K is a neighborhood of p^2.
We observe that Theorem 1 is in perfect accordance with the characterization of hypocomplexity for corank one locally integrable structure $\mathcal{V}$ over a manifold Ω (see. [1], Corollary 6.2); see also for the concepts here recalled). Indeed, as proved in [BCP] one and only one of the two cases occurs, at a given point $p^2 \in \Omega$:
(a) The Borel map for $\mathcal{V}$ at p^2 is surjective;
(b) The Borel map for $\mathcal{V}$ at p^2 is injective and $\mathcal{V}$ is hypocomplex at p^2.
We apply a smooth version of the Mergelyan theorem in order to characterize the algebras $A^{\infty}(K)$ and $M^{\infty}(K)$, when K is the closure of a bounded open subset of the complex plane.
These results hold for any compact set K; on the other hand, it is not clear whether Theorem 1 and Theorem 2 are true without the path-boundedness assumption.

2. The Algebra $A^{\infty}(K)$ and the Borel Map

2A. Basic Notation. We denote by $\mathcal{F} = \mathbb{C}[[X]]$ the algebra of formal power series in the single variable X, endowed with its natural structure as a (local) Fréchet algebra whose topology is defined by the semi-norms

$$S = \sum_{j=0}^{\infty} \sum_{r} a_j^r X^j \mapsto \mathfrak{q}_n(S) = \sum_{j=0}^{n} \sum_{r} |a_j^r|,\ n \in \mathbb{Z}_+.$$

For K be a compact subset of the complex plane. Let $C(K;\mathcal{F}) = C(K)[[X]]$, be the space of continuous functions on K valued in $\mathcal{F}$ and is also a Fréchet algebra whose topology is defined by the semi-norms

$$\mathbf{u}^r = \sum_{j=0}^{\infty} \sum_{r} u_j^r X^j \mapsto \mathbf{p}_n(\mathbf{u}^r) = \sum_{j=0}^{n} \sup_{K} \sum_{r} |u_j^r|,\ n \in \mathbb{Z}_+.$$

For $A^{\infty}(K)$ be the closure in $C(K;\mathcal{F})$ of the subalgebra

$$\mathcal{H}(K) = \left\{ \mathbf{f}_r = \sum_{j=0}^{\infty} \frac{f_r^{(j)}\big|_K}{j!} X^j \in C(K;\mathcal{F}) : f_r \in \mathcal{O}(\mathbb{C}) \right\}.$$

Note that $\gamma_K^r : \mathcal{O}(\mathbb{C}) \to \mathcal{H}(K), \gamma_K^r(f_r) = \mathbf{f}_r$ is an algebra isomorphism which is furthermore continuous, when $\mathcal{O}(\mathbb{C})$ is endowed with its natural topology of a Fréchet-Montel space. Finally, $A^{\infty}(K)(=$ the closure of the range of $\gamma_K^r)$ is a closed subalgebra of $C(K;\mathcal{F})$ and hence a Fréchet algebra itself.

2B. The Spectrum. We denote by $M(A^{\infty}(K))$ the spectrum of $A^{\infty}(K)$, that is, the set of all non trivial continuous multiplicative homomorphisms $\lambda^i : A^{\infty}(K) \to \mathbb{C}$; it is known (as first recognized by Gelfand in the setting of Banach algebras) that the association $\lambda^i \mapsto \ker \lambda^i$ gives a bijection between $M(A^{\infty}(K))$ and the set of all closed maximal ideals of $A^{\infty}(K)$. We now compute $M(A^{\infty}(K))$. Recall that $\widehat{K}$ denotes the polynomial hull of K; since $K \subset \mathbb{C}$, the maximum principle shows that the set $\widehat{K}$ is obtained from K by "filling in the holes", in other words, $\widehat{K}$ consists of K unionized with the bounded components of $\mathbb{C} \setminus K$.
If $z_0^i \in \widehat{K}$ then $\mathbf{f}_r \mapsto f_r(z_0^i)$ defines a continuous multiplicative homomorphism on $\mathcal{H}(K)$ which then extends uniquely to an element $\lambda^i_{z_0^i} \in M(A^{\infty}(K))$. Conversely let λ^i be a continuous multiplicative homomorphism on $A^{\infty}(K)$. Since γ_K^r is an algebra isomorphism it follows that $\lambda^i_\circ \gamma_K^r : \mathcal{O}(\mathbb{C}) \to \mathbb{C}$ is a continuous multiplicative homomorphism on $\mathcal{O}(\mathbb{C})$. Since the latter is a Fréchet algebra with spectrum $\mathbb{C}$ it follows that there exists $z_0^i \in \mathbb{C}$ such that $(\lambda^i \circ \gamma_K^r)(f_r) = f_r(z_0^i), f_r \in \mathcal{O}(\mathbb{C})$. But there are $\epsilon \geq 0$ and $N \in \mathbb{Z}_+$ such that

$$\sum_r \sum_i |f_r(z_0^i)| = \sum_i \sum_r |(\lambda^i \circ \gamma_K^r)(f_r)| \leq (1+\epsilon) \sum_{j=0}^{N} \sum_r \sup_K |f_r^{(j)}|,$$

since λ^i is continuous on $A^\infty(K)$. By Cauchy estimates for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\sum_r \sum_i |f_r(z_0^i)| \leq C_\varepsilon \sup_{K_\varepsilon} \sum_r |f_r|, \ f_r \in \mathcal{O}(\mathbb{C}),$$

where $K_\varepsilon = \{z^i : \text{dist}\,(z^i, K) \leq \varepsilon\}$. Replacing f_r by $f_r^{p^2}$ and letting $p^2 \to \infty$ shows that the preceding inequality holds with $C_\varepsilon = 1$ for every $\varepsilon > 0$ and thus $|f_r(z_0^i)| \leq \sup_K |f_r|, f_r \in \mathcal{O}(\mathbb{C})$. This is equivalent to saying that $z_0^i \in \widehat{K}$. We have proved: the map $\widehat{K} \to M(A^\infty(K)), z_0^i \mapsto \lambda_{z_0^i}^i$, is a bijection. Thus, the spectrum of $A^\infty(K)$ equals $\widehat{K}$.

2C. Other Dense Subspaces. By Runge's theorem every element in $\mathcal{O}(\widehat{K})$ can be approximated uniformly near $\widehat{K}$ by a sequence of entire functions. Hence it follows that

$$\mathcal{H}^{\cdot}(K) = \left\{ \mathbf{f}_r = \sum_{j=0}^{\infty} \sum_r \frac{f_r^{(j)}\big|_K}{j!} X^j \in C(K;\mathcal{F}) : f_r \in \mathcal{O}(\widehat{K}) \right\}$$

is also a dense subalgebra of $A^\infty(K)$ and that $\mathcal{H}(K) \subset \mathcal{H}^{\cdot}(K)$. Notice also that we have an algebra isomorphism $\dot{\gamma}_K^r : \mathcal{O}(\widehat{K}) \to \mathcal{H}^{\cdot}(K)$ which is continuous when $\mathcal{O}(\widehat{K})$ is endowed with its natural DFS topology by a similar argument as above in 2B, we using the homomorphism $\dot{\gamma}_K^r$, and give a proof that the spectrum of $A^\infty(K)$ equals $\widehat{K}$. This idea will be exploited in Section $5A$. Finally, we also note that the polynomials are a dense subalgebra of $A^\infty(K)$.

2D. Mapping Properties. For $F_r : \mathbb{C} \to \mathbb{C}$ be a holomorphic map and let K be as in A. Then $F_r^* : \mathcal{H}(F_r(K)) \to \mathcal{H}(K)$ defined by

$$\sum_{j=0}^{\infty} \sum_r \left(f_r^{(j)}\big|_K / j!\right) X^j \longrightarrow \sum_{j=0}^{\infty} \sum_r \left\{ \frac{(f_r \circ F_r)^{(j)}\big|_K}{j!} \right\} X^j, \ f_r \in \mathcal{O}(\mathbb{C}),$$

is an algebra homomorphism which is continuous by the Faà di Bruno formula and hence it can be extended as a continuous algebra homomorphism $F_r^* : A^\infty(F_r(K)) \to A^\infty(K)$.

2E. The Borel Map. Note that if $z_0^i \in \mathbb{C}$ we have $A^\infty(\{z_0^i\}) = \mathbb{C}[\,[X]\,]$. Indeed, any series $\sum_{j=0}^{\infty} \sum_i \sum_r a_j^r (z^i - z_0^i)^j$ is the limit of its truncations

$$P_n = \sum_{j=0}^{n} \sum_r \sum_i a_j^r (z^i - z_0^i)^j,$$

and the seminorms actually agree. This is a large distinction with $A(K)$ (closure of $\mathcal{O}(\mathbb{C})|_K$ in $C(K)$) since $A(\{0\}) = \mathbb{C}$, and suggests that in the study of $A^\infty(K)$, instead of multiplicative characters of the algebra $A(K)$, that is, homomorphisms valued in $\mathbb{C}$, one needs to study homomorphisms $A^\infty(K) \to \mathcal{F}$ in order to algebraically capture properties of points in K with respect to the structure of $A^\infty(K)$.

This leads to the following definition.

Definition 2. Assume that $0 \in K$. We shall refer to the continuous algebra homomorphism
$\mathfrak{b}_K : A^\infty(K) \to \mathcal{F}, \ \mathfrak{b}_K(\mathbf{u}^r) = \mathbf{u}^r(0)$
as the Borel map of K at the origin.

By subsection $2B$ when K is not a singleton then $A^\infty(K)$ is not a local algebra (for the spectrum of a local algebra has only one element). Since $\mathcal{F}$ is a local algebra we conclude that the Borel map for K at the origin is never bijective we have the following (see [12]).

Example 1. Suppose that K is a real compact set with $0 \in K$. Then the Borel map of K (at the origin) is surjective. Indeed if $s \in \mathcal{F}$ by the classical Borel theorem there is $g_r \in C^\infty(\mathbb{R})$ whose formal Taylor expansion at the origin is equal to s. Fix a compactly supported, smooth function ψ on $\mathbb{R}$ which is equal to one in an open neighborhood of K and form

$$G_j(z^i) = \sum_j (j/\pi)^{\frac{1}{2}} \int_{\mathbb{R}} \sum_i \sum_r \exp\{-j(z^i - y)^2\} \psi(y) g_r(y) \mathrm{d}y, \ z^i \in \mathbb{C}.$$

Then $\{G_j\} \subset \mathcal{O}(\mathbb{C})$ and it is well known that $G_j^{(k)} \to (\psi g_r)^{(k)}$ uniformly in $\mathbb{R}$, for every $k \geq 0$. In particular $G_j^{(k)}\big|_K \to g_r^{(k)}\big|_K$ uniformly in K for every $k \geq 0$ and consequently the formal power series $G_j := \sum_{k=0}^{\infty} \sum_r \left(g_r^{(k)}\big|_K\right) X^k/k!$ belongs to $A^\infty(K)$ and $\mathbf{b}_K(G_j) = \mathbf{s}$.

Remark 1. If $K + \epsilon \subset K \subset \mathbb{C}$ are compact sets, both containing the origin, then there is a natural algebra homomorphism $r_{K,K+\epsilon}: A^\infty(K) \to A^\infty(K+\epsilon)$ induced by restriction of functions and it is clear that $\mathfrak{b}_K = \mathfrak{b}_{K+\epsilon} \circ r_{K,K+\epsilon}$.

2F. Surjectivity of the Borel Map: Necessary Criterion. From now on we shall always assume that $0 \in K$. We now apply some standard functional analytic arguments to derive a necessary condition for the (non) surjectivity of the Borel map:

Theorem 3 (see [12]). $\mathfrak{b}_K$ is not surjective if and only if there are a sequence $\{P_j\} \subset \mathbb{C}[X]$ and $M \in \mathbb{Z}_+$ such that degree $P_j \longrightarrow \infty$ and

$$|(P_j(\partial)h)(0)| \leq \sum_{k=0}^{M} \sup_K |h^{(k)}| = \sum_{k=0}^{M} \sup_{\hat{K}} |h^{(k)}|,$$

for $h \in \mathcal{O}(\hat{K})$ and $j = 1,2, \ldots$

This property implies that certain combinations of arbitrarily high derivatives of a holomorphic function at 0 can be estimated by the sup of derivatives of order $\leq M$ on $\hat{K}$. Note that the Borel map is in particular not surjective if $0 \in K^\circ$.

Proof. Suppose that $\mathfrak{b}_K$ is not surjective. Since the image of $\mathfrak{b}_K$ contains $\mathbb{C}[X]$, it is dense in $\mathbb{C}[[X]]$; in particular, $\mathfrak{b}_K$ is not a homomorphism. By the homomorphism theorem [11, p.18] the image of the transpose
${}^t\mathbf{b}_K: \mathbb{C}[X] \longrightarrow A^\infty(K)'$
is not strongly sequentially closed. It follows that there is a sequence $\{Q_k\} \subset \mathbb{C}[X]$ such that $\Gamma = \{{}^t\mathfrak{b}_K(Q_k): k \geq 1\}$ is strongly bounded and hence equicontinuous but $\{Q_k\}$ has no convergent subsequence. Since $\mathbb{C}[X]$ is a Montel space (with its usual LF-topology) it follows that $\{Q_k\}$ is not bounded. Notice that there is no $L \geq 0$ such that $\{Q_k\} \subset \mathbb{C}_{(L)}[X]$, the space of polynomials of degree $\leq L$. Indeed since the latter is finite dimensional, ${}^t\mathfrak{b}_K: \mathbb{C}_{(L)}[X] \to A^\infty(K)'$ is a homomorphism and the fact that Γ is strongly bounded would imply that $\{Q_k\}$ is bounded, a contradiction. In conclusion we can extract a subsequence $k_j < k_{j+1}$ such that the degree of Q_{k_j} is $> j$ for every j. Setting $P_j = Q_{k_j}$ proves the existence of the sequence $\{P_j\}$ with the required properties.
Conversely the existence of the sequence $\{P_j\}$ gives the existence of a weakly unbounded sequence in the dual of $\mathbb{C}[[X]]$ whose image by ${}^t\mathfrak{b}_K$ is a weakly bounded set in the dual of $A^\infty(K)$. Since ${}^t\mathfrak{b}_K$ is injective the map
$({}^t\mathbf{b}_K)^{-1}: {}^t\mathbf{b}_K(\mathbb{C}[[X]]) \longrightarrow \mathbb{C}[[X]]$
is well defined and not continuous for the weak topologies. Then $\mathfrak{b}_K$ is not a homomorphism [11, p.18] and hence it is not surjective.

3. The Main Result – Beginning

As shown before, if $\hat{K}$ is a neighborhood of the origin then the image of the Borel map of K at the origin is contained in the space of all convergent power series and hence the Borel map is injective but not surjective. We shall prove that under a suitable geometrical assumption on K (path-boundedness, see Definition 1) the following stronger properties hold: $\mathfrak{b}_K$ is alternatively either injective or surjective and moreover $\mathfrak{b}_K$ is injective if and only if $\hat{K}$ is a neighborhood of the origin.

A particular but important class of compact sets which satisfy the property described in Definition 1 is the class of the subanalytic compact sets. Before we prove the general result we present a proof for the subanalytic case, which we believe is interesting in its own right and also turns out to be more suited for questions in higher dimensions.

3A. Surjectivity of the Borel Map: Characterization for Subanalytic Sets. We begin by proving (see [12]):

Theorem 4. Assume $K \subset \mathbb{C}$ is compact and subanalytic and that $\mathbf{b}_K$ is not surjective. Then $\hat{K}$ is a neighborhood of the origin.

Proof. According to Theorem 3 there are a sequence of polynomials $\{P_j\} \subset \mathbb{C}[X]$, with degree of $P_j \doteq \mu_j \to \infty$ and a non negative integer M such that

$$|P_j(\partial)h(0)| \leq \sum_{\ell=0}^{M} \sup_K |h^{(\ell)}|,\ h \in \mathcal{O}(\hat{K}).$$

We prove our result by contradiction. Assume that $\hat{K}$ is not a neighborhood of the origin. Since $\hat{K}$, as the union of K with the bounded components of $\mathbb{C} \setminus K$, is itself subanalytic (see [10], Prop. 3.2 and Corollary 3.7.10), we conclude that $\mathbb{C} \setminus \hat{K}$ is also subanalytic. Since the origin is in the closure of $\mathbb{C} \setminus \hat{K}$, using ([10], Prop. 3.9) we can find a real-analytic map $\gamma^r:]-1,1[\to \mathbb{C}$ such that

$$\gamma^r(0) = 0, \qquad \gamma^r(s) \notin \hat{K}, s \neq 0,\ |\gamma^r| \leq 1.$$

Notice that there are $\epsilon \geq 0$ and $p^2 \in \mathbb{N}$ such that

$$|\gamma^r(s)| \leq (1+\epsilon)|s|^{p^2}, |s| < 1.$$

Now given $s \neq 0$ the function $h_s(z^i) = (\gamma^r(s) - z^i)^{-1}$ belongs to $\mathcal{O}(\widehat{K})$ and it satisfies $h_s^{(k)}(0) = k!\,\gamma^r(s)^{-(k+1)}$ for every $k \geq 0$. If we write

$$P_j(X) = \sum_j \sum_{k=0}^{\mu_j} A_{jk} X^k / k!,\ A_{j,\mu_j} \neq 0,$$

then (1) gives

$$\begin{aligned} \left| \sum_j \sum_{k=0}^{\mu_j} \sum_r A_{jk} \gamma^r(s)^{-(k+1)} \right| &\leq \sum_{\ell \leq M} \sup_{\widehat{K}} \left| h_s^{(\ell)} \right| \\ &\leq C_M \sum_{\ell \leq M} \sum_r \text{dist}\ (\gamma^r(s), \widehat{K})^{-(\ell+1)},\ j \in \mathbb{Z}_+, s \neq 0, \end{aligned}$$

and, consequently,

$$\text{dist}\ (\gamma^r(s), \widehat{K})^{M+1} \left| \sum_j \sum_{k=0}^{\mu_j} \sum_r A_{jk} \gamma^r(s)^{\mu_j - k} \right| \leq C'_M \sum_r \sum_j |\gamma^r(s)|^{\mu_j + 1},$$

We observe that there exists a sequence $\{\varepsilon_j\} \subset]0,1[$ such that

$$\left| \sum_j \sum_{k=0}^{\mu_j} \sum_r A_{jk} \gamma^r(s)^{\mu_j - k} \right| \geq \sum_j \left| A_{j,\mu_j} \right| - \sum_j \sum_{k=0}^{\mu_j - 1} \sum_r |A_{jk}| |\gamma^r(s)|^{\mu_j - k} \geq \left| A_{j,\mu_j} \right| / 2$$

if $|s| < \varepsilon_j$, and hence it follows from (2) and (3) that
$\text{dist}\ (\gamma^r(s), \widehat{K})^{M+1} \leq C''_{j,M} |s|^{p^2(\mu_j + 1)},\ |s| < \varepsilon_j.$

The function $s \mapsto \text{dist}\ (\gamma^r(s), \widehat{K})$ is subanalytic (see $e.g.$ $[3]$, Remark 3.11) and does not vanish for $s \neq 0$. We then apply the Lojasiewicz inequality ($[3]$, Theorem 6.4): there are constants $c, r > 0$ such that

$$c|s|^r \leq \text{dist}\ (\gamma^r(s), \widehat{K}),\ |s| < 1.$$

We then obtain

$$|s|^{r(M+1)} \leq C'''_{j,M} |s|^{p^2(\mu_j + 1)},\ |s| < \varepsilon_j.$$

If we fix j such that $p^2(\mu_j + 1) > r(M+1)$ (which is possible because $\mu_j \to \infty$) we obtain a contradiction. This completes the proof.
As a consequence we derive (see [12]):
Corollary 1. Let $K \subset \mathbb{C}$ be compact and subanalytic. Then one and only one of the properties hold:
(i) $\mathfrak{b}_K$ is surjective;
(ii) $\mathfrak{b}_K$ is injective and $\widehat{K}$ is a neighborhood of the origin.

4. On the Notion of Capacity

4A. Capacity and Relative Peaking. We assume that $K \subset \mathbb{C}$ is compact and polynomially convex, and that $0 \in \partial K$. We first recall the notion of analytic capacity and then use it to show that we can "expose" the point 0 by a sequence of functions in $\mathcal{O}(K)$ with large derivatives (at 0). We will largely follow Gamelin's book ($[8]$, Chapter $VIII$, Theorem 4.1). In particular, we use the definition of analytic capacity and employ the same notation as there: if $L \subset \mathbb{C}$ is a compact set then $\Omega(L)$ denotes the connected component of $S^2 \setminus L$ containing ∞ (where $S^2 = \mathbb{C} \cup \{\infty\}$) and $\gamma^r(L)$ denotes the analytic capacity. We will need in particular the following results, where Δ denotes the unit disc centered at the origin:

Theorem 5. ([8], $VIII$, Thm. 1.4) Assume L connected. Let $g_r: \Omega(L) \to \Delta$ be the conformal map such that $g_r(\infty) = 0$ and $g_r'(\infty) > 0$. Then $\gamma^r(L) = g_r'(\infty)$.

Theorem 6. ([8], $VIII$, Thm. 2.1) Assume L connected. Then

$$\gamma^r(L) \le \text{diam}\,(L) \le 4\gamma^r(L).$$

We can now state (see [12])

Proposition 1. Suppose that K is polynomially convex and that the origin belongs to the boundary of K. Then there is a sequence $\{(g_r)_n\} \subset \mathcal{O}(K)$ such that $|(g_r)_n| \le 1 + \epsilon$ on K for every $n \in \mathbb{N}$ and $(g_r)_n'(0) \to \infty$.

Proof. Let $\{r_n\}$ be a positive sequence of radii such that $r_n \to 0$ as $n \to \infty$, and let Δ_{r_n} be the disc of center 0 and radius r_n. For any $n \in \mathbb{N}$, all the connected components of $\Delta_{r_n} \setminus K$ intersect $\partial\Delta_{r_n}$; otherwise there would be one contained in Δ_{r_n}, hence $\mathbb{C} \setminus K$ would have a bounded connected component, against the assumption that K is polynomially convex.

For any n choose a point $p_n^2 \in \Delta_{r_n} \setminus K$ with $|p_n^2| < r_n/2$; by the previous argument there exists a smooth curve $\gamma_n^r \subset \Delta_{r_n} \setminus K$ that connects p_n^2 to a point in $b\Delta_{r_n}$. Thus the diameter of γ_n^r is at least $r_n/2$, and there exists a compact, connected subset $E_n \subset \gamma_n^r$ with diameter $r_n/2$.

Let $(f_r)_n: \Omega(E_n) \to \Delta$ be the conformal map such that $(f_r)_n(\infty) = 0$ and $(f_r)_n'(\infty) > 0$.

By Theorems 5 and 6 and above we have that $(f_r)_n'(\infty) = \gamma^r(E_n)$ and since

$$\frac{\text{diam}\,(E_n)}{4} \le \gamma^r(E_n) \le \text{diam}(E_n),$$

it follows that

$$\frac{r_n}{8} \le (f_r)_n'(\infty) \le \frac{r_n}{2}.$$

We further define $(g_r)_n: \Omega(E_n) \to \mathbb{C}$ as $(g_r)_n(z^i) = z^i (f_r)_n(z^i)/(f_r)_n'(\infty)$. For all $z^i \in \Delta_{r_n} \setminus E_n$, applying (4) we can write

$$\sum_r \sum_j |(g_r)_n(z^i)| = \sum_r \sum_j \frac{|z^i||(f_r)_n(z^i)|}{(f_r)_n'(\infty)} \le \frac{r_n \cdot 1}{r_n/8} = 8$$

and by the maximum principle it follows that $|(g_r)_n| \le 8$ on $\Omega(E_n)$; in particular $|(g_r)_n| \le 8$ on the set $K \subset \Omega(E_n)$.

Since $(g_r)_n$ is holomorphic on a neighborhood of K it represents an element of $A^\infty(K)$, hence $\{(g_r)_n\}$ is a sequence in $A^\infty(K)$ which is bounded in the uniform norm. Since

$$\sum_r |(g_r)_n'(0)| = \sum_r \frac{|(f_r)_n(0)|}{(f_r)_n'(\infty)} \ge \frac{2|(f_r)_n(0)|}{r_n},$$

in order to finish the proof, it suffices to show that $\{|(f_r)_n(0)| : n \in \mathbb{N}\}$ is bounded below. Indeed we claim that

$$\sum_r |(f_r)_n(0)| \ge \frac{1}{32},\ n \in \mathbb{N}.$$

To prove this claim we define $k_n: \Delta \to \mathbb{C}$ as $k_n(z^i) = (f_r)_n(r_n/z^i)$. Since $(f_r)_n$ is injective on $S^2 \setminus \Delta_{r_n}$ (indeed $(f_r)_n$ is the inverse of the Riemann map $\Delta \to \Omega(E_n)$), it follows that k_n is injective on Δ. Moreover, since $(f_r)_n(z^i) = (f_r)_n'(\infty)/z^i + O(1/z^{2i})$ on $S^2 \setminus \Delta_{r_n}$, we have $k_n(z^i) = \left(\frac{(f_r)_n'(\infty)}{r_n}\right) z^i + O(z^{2i})$, hence $k_n(0) = 0$ and $k_n'(0) = (f_r)_n'(\infty)/r_n$. Using the estimates in (4) we get $|k_n'(0)| = k_n'(0) \ge 1/8$. By Koebe's theorem it follows that $k_n(\Delta) \supset \Delta_{1/32}$. By construction $k_n(\Delta) = (f_r)_n(S^2 \setminus \Delta_{r_n})$, thus we also have $(f_r)_n(S^2 \setminus \Delta_{r_n}) \supset \Delta_{1/32}$. The injectivity of $(f_r)_n$ in turn implies that $(f_r)_n(0) \notin \Delta_{1/32}$, i.e. $|(f_r)_n(0)| \ge 1/32$.

5. The Main Result – Conclusion

5A. The Notion of Path-Boundedness. We say that K is path bounded if there exists a $\epsilon \ge 0$ such that for every open set $U \subset K$, any pair of points $p^2, q^2 \in K$ can be connected by a rectifiable curve (in U) of length not exceeding $1 + \epsilon$. Note that one can replace rectifiable curves by polygonal paths in this definition. Before we proceed, we give some examples (see [12]):

Example 2. Every compact, connected subanalytic set $K \subset \mathbb{C}$ is path-bounded.

Indeed ([9], Section 6) shows that even more is true: Given such a K there are constants $1+\epsilon, \alpha$ such that for any pair of points $p^2, q^2 \in K$ there is a rectifiable curve γ^r contained in K with length $\leq (1+\epsilon)|p^2 - q^2|^\alpha$.

Example 3. Of course, the notion of path-boundedness is much weaker than subanalyticity. We include an example for the reader's convenience: Let $K \subset \mathbb{C}$ be defined as

$$K = [-1,0] \times [-1,1] \cup [0,1] \times [-1,0] \cup \bigcup_{j=1}^{\infty} \left[\frac{1}{j^2+1}, \frac{1}{j^2}\right] \times [0,1].$$

Then K is compact and path-bounded (any two points $p^2, q^2 \in K$ can be joined by a rectifiable curve of length less than 4 contained in K) and $\mathbb{C} \setminus K$ is connected, implying that K is polynomially convex. On the other hand, K is not subanalytic, since for any point $p^2 = it (0 \leq t < 1)$ we can find a neighborhood U of p^2 in $\mathbb{C}$ such that $U \setminus K$ has infinitely many connected components.

Example 4. Let us give an example of a set which is not path-bounded. For this, we consider the set $K = \{x + i f_r(x) \in \mathbb{C}: x \in (0,1]\} \cup \{0\} \times [0,1]$ where f_r is defined by

$$f_r(x) = 1 - \frac{1}{2j(j+1)}\left|x - \frac{2j+1}{2j(j+1)}\right|, \ \frac{1}{j+1} \leq x \leq \frac{1}{j}.$$

Choose then a sequence of open sets U_j defined by

$$U_j = \left(-1, \frac{1}{j}\right) \times (-2,2) \cup \left\{x + iy \in \mathbb{C}: \frac{1}{j} \leq x < 1, |y - f_r(x)| < \frac{1}{16}\right\} \cup D_{1/16}(1).$$

One checks that the path of smallest length connecting $1/j$ with 1 contained in U_j has length exceeding j, and thus, K is not path-bounded.
Note that in the previous example the Borel map at the origin is nevertheless onto, since K is contained in $[0,1]^2$.
Taking polynomial hulls preserves path-boundedness, as the next Lemma shows (see [23]).
Lemma 1. If K is path-bounded then the same is true for $\widehat{K}$.
Proof. Let $\epsilon \geq 0$ be such that the property described in the preceding definition holds and let $C_1 := 3(1+\epsilon)$, where $d > 0$ is the diameter of $\widehat{K}$. Let now $U \subset \mathbb{C}$ be an open neighborhood of $\widehat{K}$ and let $p^2, q^2 \in \widehat{K}$. If $p^2, q^2 \in K$, since U is in particular also a neighborhood is K, then p^2 and q^2 can be joined by a rectifiable curve γ^r contained in U and with length $\leq 1+\epsilon$. Now suppose that $p^2 \in \widehat{K} \setminus K$. Then p^2 belongs to one of the open, bounded components V of $\mathbb{C} \setminus K$. Let p_0^2 be the nearest point to p^2 belonging to $\partial V \subset K$.
If γ_0^r is a rectifiable curve joining p_0^2 and q^2 and contained in U with length $\leq 1+\epsilon$ then $[p^2, p_0^2] \vee \gamma_0^r$ is a rectifiable curve joining p^2 and q^2 contained in U and with length $\leq 2(1+\epsilon) < C_1$. An analogous argument treats the case when both p^2 and q^2 do not belong to K, in which case the length of the curve obtained is $\leq 3(1+\epsilon) = C_1$.
The importance of the concept of path-boundedness for the Borel map is explained by the next result (see [12]):
Proposition 2. Assume that K is a path-bounded polynomially convex compact set and that the origin belongs to the boundary of K. Given $m \in \mathbb{N}$, there exists a sequence $\{h_n\} \in \mathcal{O}(K)$ such that

$$\sup_{z^i \in K} \sum_i \left|h_n^{(j)}(z^i)\right| < 1, \ j < m,$$

but $\left|h_n^{(m)}(0)\right| \to +\infty$ as $n \to \infty$.

In other words, for this sequence $h_n, \mathrm{p}_{m-1}(\gamma_K^r(h_n))$ is bounded but $\left|h_n^{(m)}(0)\right| \to +\infty$ as $n \to \infty$.
Proof. We apply induction, the case $m = 1$ being provided by the sequence $(g_r)_n$ given by Proposition 1. For the inductive step, suppose that a sequence $\{h_n\}$ has been constructed for $m-1$, and furthermore suppose that h_n is defined on the set $\Omega(E_n)$ (with E_n from the proof of Proposition 1). Since $\Omega(E_n)$ is simply connected, for all $n \in \mathbb{N}$ there exists a function H_n holomorphic on $\Omega(E_n)$ such that $H_n' = h_n$ and $H_n(0) = 0$. Let $q^2 \in K$: by assumption there exists a curve $\gamma^r \subset \Omega(E_n)$ of length at most $1+\epsilon$ joining 0 and q^2. It follows that $|H_n(q^2)| \leq (1+\epsilon) p_1^2(\gamma_K^r(h_n)) \leq (1+\epsilon) p_{m-2}^2(\gamma_K^{*r}(h_n))$.
Since q^2 is arbitrary we conclude that $p_0^2(\gamma_K^r(H_n)) \leq (1+\epsilon) p_{m-2}^2(\gamma_K^r(h_n))$. The claim then follows from the facts that the sequence

$$\begin{aligned} \mathrm{p}_{m-1}(\gamma_K^r(H_n)) &\leq max \sum_r \{\mathrm{p}_0(\gamma_K^r(H_n)), \mathrm{p}_{m-2}(\gamma_K^r(H_n'))\} \\ &\leq max \sum_r \{(1+\epsilon)\mathrm{p}_{m-2}(\gamma_K^r(h_n)), \mathrm{p}_{m-2}(\gamma_K^r(h_n))\} \end{aligned}$$

is bounded and that $\left|H_n^{(m)}(0)\right| = \left|h_n^{(m-1)}(0)\right| \to +\infty$ as $n \to \infty$.

5B. Surjectivity of the Borel Map: the General Case. We now have to prove the characterization of Theorem 4 for general path-bounded sets.

Theorem 7 (see [12]). Assume that K is a path-bounded compact set and assume also that $\mathfrak{b}_K$ is not surjective. Then $\widehat{K}$ is a neighborhood of the origin.

Proof. Since $\mathfrak{b}_K$ is not surjective we can apply Theorem 3 and conclude the existence of sequences $\{m_j\} \subset \mathbb{Z}_+, \{C_j\} \subset]0, \infty[$, with $m_j \to \infty$, such that $\left|h^{(m_j)}(0)\right| \leq C_j \mathbf{p}_{m_j-1}(\gamma_K^r(h))$ for all $h \in \mathcal{O}(\widehat{K})$. Now by Lemma 1 it follows that $\widehat{K}$ is also pathbounded and hence by Proposition 2 applied to, say, $m = m_1$ we conclude that the origin cannot be a boundary point of $\widehat{K}$.

6. The Algebra $M^\infty(K)$

To study functions on K instead of $\widehat{K}$, one has to use a richer algebra (making its spectrum smaller). Indeed, if we replace the space of entire functions in $\mathbb{C}$ by the space of meromorphic functions on $\mathbb{C}$ with poles outside K we obtain a paralel theory in which $\widehat{K}$ will no longer plays a role. Since the arguments are similar we state the main results.

6A. The Algebra and its Spectrum. We denote by $M^\infty(K)$ the closure in $C(K;\mathcal{F})$ of the subalgebra

$\mathcal{H}^\dagger(K) = \left\{\mathbf{f}_r = \sum_{j=0}^\infty \left(\sum_r f_r^{(j)}\big|_K / j!\right) X^j \in C(K;\mathcal{F}) : f_r\right.$ meromorphic with no poles in $\left.K\right\}$.

Notice that $\mathcal{H}(K) \subset \mathcal{H}^\dagger(K)$ and hence $A^\infty(K)$ is a subalgebra of $M^\infty(K)$. By Runge's theorem any element in $\mathcal{O}(K)$ can be approximated near K by a sequence of meromorphic functions in $\mathbb{C}$ with prescribed location of their poles.

Hence $M^\infty(K)$ is equal to the closure in $C(K;\mathcal{F})$ of the subalgebra

$$\mathcal{H}^\ddagger(K) = \left\{\mathbf{f}_r = \sum_{j=0}^\infty \sum_r \left(f_r^{(j)}\big|_K / j!\right) X^j \in C(K;\mathcal{F}) : f_r \in \mathcal{O}(K)\right\}.$$

Proposition 3 (see [12]). The spectrum of $M^\infty(K)$ is the compact set K itself. In particular, K is polynomially convex if and only if $A^\infty(K) = M^\infty(K)$.

Proof. Consider the algebra isomorphism $(\gamma_K^r)^\ddagger : \mathcal{O}(K) \to \mathcal{H}^\ddagger(K)$, $(\gamma_K^r)^\ddagger(f_r) = \mathbf{f}_r$. It is clear that $(\gamma_K^r)^\ddagger$ is continuous when $\mathcal{O}(K)$ is endowed with its DFS topology. Let $\lambda^i : M^\infty(K) \to \mathbb{C}$ be a continuous multiplicative homomorphism. Then $\lambda^i \circ (\gamma_K^r)^\ddagger$ is a continuous multiplicative functional on $\mathcal{O}(K)$. By the definition of the DFS topology on $\mathcal{O}(K)$ given any $U \supset K$ open the map

$\sigma_U := \mathcal{O}(U) \longrightarrow \mathcal{O}(K) \overset{\lambda^i \circ \gamma_K^\ddagger}{\longrightarrow} \mathbb{C}$

is a continuous multiplicative homomorphism on $\mathcal{O}(U)$ and hence it is equal to the evaluation at some point $z_0^i \in U$. If $K \subset V \subset U$ is also open then $\sigma_V = \sigma_U|_V$ and hence z_0^i must a fortiori belong to V. Letting $U \to K$ we conclude that $z_0^i \in K$, which proves the first assertion in the statement. For the second it suffices to notice that when K is polynomially convex then $\mathcal{O}(\widehat{K}) = \mathcal{O}(K)$.

6 B. The Borel Map. If we assume that $0 \in K$ (as we shall do from now on), we can also define the Borel map relative to $M^\infty(K)$ as being

$\mathbf{b}_K^\dagger(\mathbf{u}^r) = \mathbf{u}^r(0)$, $\mathbf{u}^r \in M^\infty(K)$.

It is clear that $\mathfrak{b}_K^\dagger = \mathfrak{b}_K$ in $A^\infty(K)$.

The following result can be obtained by the same argument as in the proof of Theorem 3.

Theorem 8. $\mathfrak{b}_K^\dagger$ is not surjective if and only there are a sequence $\{P_j\} \subset \mathbb{C}[X]$ and $M \in \mathbb{Z}_+$such that degree $P_j \longrightarrow \infty$ and

$\left|\left(P_j(\partial)h\right)(0)\right| \leq \sum_{\alpha \leq M} \sup_K |\partial^\alpha h|$,

for $h \in \mathcal{O}(K)$ and $j \in \mathbb{N}$.

6C. Surjectivity Revisited for $M^\infty(K)$. The condition needed on K is a bit stronger in this setting:

Theorem 9. Let K be path-bounded and assume that K^c has only finitely many connected components in a neighbourhood of 0. If $\mathfrak{b}_K^\dagger$ is not surjective then K is a neighborhood of the origin.

The proof is the same as that of Theorem 7 taking into account the following remark:

Remark 2. Let K be a path-bounded compact set, $0 \in \partial K$, and suppose that $\mathbb{C} \setminus K$ has finitely many connected components. For any $m \in \mathbb{N}$, there exists a sequence $\{h_n\} \in \mathcal{O}(K)$ such that $p_{m-1}^2\left((\gamma_K^r)^\ddagger(h_n)\right)$ is bounded but $\left|h_n^{(m)}(0)\right| \to +\infty$ as $n \to \infty$.

Indeed, let $1 + \epsilon$ be the minimum of the diameters of the finitely many bounded connected components of $\mathbb{C} \setminus K$ lying in Δ_{r_0}. Then, for $0 < r_0 < \frac{1+\epsilon}{2}$ all the connected components of $\Delta_{r_0} \setminus K$ intersect $\partial \Delta_{r_0}$. This allows to construct a sequence $(g_r)_n \in \mathcal{O}(K)$

using exactly the same procedure as in Proposition 1, which proves the claim for $m = 1$. The inductive argument used to extend the claim to all m is unchanged: once again, the functions H_n obtained as primitives of h_n on $\Omega(E_n)$ define holomorphic functions in a neighborhood of K.

7. Final Remarks

7A. Closures of Simply Connected Open Sets. If $\Omega \subset \mathbb{C}$ is a bounded open set then $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$ is a Fréchet algebra whose topology is defined by the norms

$$f_r \mapsto \sum_{j=0}^{n} \frac{1}{j!} \sup_{\Omega} \sum_r \left|f_r^{(j)}\right|, \; n \in \mathbb{Z}_+.$$

Here, we denote by $\mathcal{C}^\infty(\bar{\Omega})$ the set of restrictions of $\mathcal{C}^\infty(\mathbb{C})$ (or equivalently, of $\mathcal{C}_c^\infty(\mathbb{C})$) to $\bar{\Omega}$.
We now show (see [12]):

Theorem 10. Let $\Omega \subset \mathbb{C}$ be a bounded open set. If $\mathbb{C} \setminus \bar{\Omega}$ is connected then
$A^\infty(\bar{\Omega}) = \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$
as Fréchet algebras.

Proof. We must show that for every $f_r \in \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$ there is a sequence $\{(f_r)_n\} \subset \mathcal{O}(\mathbb{C})$ such that $(f_r)_n^{(j)} \to f_r^{(j)}$ uniformly in $\bar{\Omega}$, for every $j \geq 0$. This is of course the smooth version of the classical Mergelyan's theorem whose proof, although probably known, we present for the sake of completeness.
Let then f_r be as above and take $F_r \in \mathcal{C}_c^\infty(\mathbb{C})$ such that $F_r = f_r$ in Ω. Let $d(z^i)$ denote the distance function from $z^i \in \mathbb{C}$ to $\bar{\Omega}$. For $\delta > 0$, let
$\Omega_\delta := \{z^i \in \mathbb{C}: d(z^i) < \delta\}$
and form

$$u_\delta^r(z^i) = \frac{1}{\pi} \iint_{\Omega_\delta} \sum_i \sum_r \frac{1}{z^i - w} \frac{\partial F_r}{\partial \bar{z}^i}(w) \mathrm{dm}(w), \; z^i \in \Omega_\delta.$$

Here m denote the Lebesque measure in the plane. Notice that $u^r \in \mathcal{C}^\infty(\Omega_\delta)$ and that $F_r - u_\delta^r \in \mathcal{O}(\Omega_\delta)$. Furthermore, since $\partial F_r / \partial \bar{z}^i$ vanishes to infinite order at $\bar{\Omega}$ there is a sequence of positive constants $C_m > 0, m = 0,1, \dots$ such that

$$\left| \sum_r \sum_i \frac{\partial F_r}{\partial \bar{z}^i}(z^i) \right| \leq C_m d(z^i)^m, \; z^i \in \mathbb{C}.$$

Hence if $j \geq 0$ and if we take $m > j$ we can estimate

$$\left| \sum_i \sum_j \sum_r \frac{\partial^j u_\delta^r}{\partial z^{ji}}(z^i) \right| \leq \frac{C_m}{\pi} \iint_{\Omega_\delta} \sum_i \frac{d(w)^m}{d(w)^j} \mathrm{dm}(w) \leq \frac{C_m \delta^{m-j}}{\pi} \mathrm{m}(\Omega_\delta), \; z^i \in \bar{\Omega}.$$

Since also $\partial u_\delta^r / \partial \bar{z}^i = \partial F_r / \partial \bar{z}^i$ we then conclude that
$u_\delta^r \to 0$ as $\delta \to 0$ in $\mathcal{C}^\infty(\bar{\Omega})$.

Finally, we apply Runge's theorem: given $\delta > 0$ there is a sequence $\{(g_r)_{n,\delta}\} \subset \mathcal{O}(\mathbb{C})$ such that $(g_r)_{n,\delta} \to (F_r - u_\delta^r)$ as $n \to \infty$ in $\mathcal{O}(\Omega_{\delta/2})$ and hence in $\mathcal{C}^\infty(\bar{\Omega})$ by Cauchy estimates. To conclude the argument it is convenient to introduce a translation invariant distance function ρ which defines the Fréchet topology of $\mathcal{C}^\infty(\bar{\Omega})$. Then

$$\rho\big((g_r)_{n,\delta}, f_r\big) = \rho\big((g_r)_{n,\delta}, F_r\big) \leq \rho\big((g_r)_{n,\delta}, F_r - u_\delta^r\big) + \rho(u_\delta^r, 0).$$

Given $\varepsilon > 0$ we first choose $\delta > 0$ such that $\rho(u_\delta^r, 0) < \varepsilon/2$ and after we choose $n \in \mathbb{N}$ such that $\rho\big((g_r)_{n,\delta}, F_r - u_\delta^r\big) < \varepsilon/2$. Hence for every $\varepsilon > 0$ we can find $g_r \in \mathcal{O}(\mathbb{C})$ such that $\rho(g_r, f_r) < \varepsilon$, which is the conclusion of the theorem.

7B. Closures of Open Sets: The General Case. As in Section 5 a small change in the preceding argument also gives:
Theorem 11 [12]. Let $\Omega \subset \mathbb{C}$ be a bounded open set. Then
$M^\infty(\bar{\Omega}) = \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega})$
as Fréchet algebras.

8. References

1. Barostichi RF, Cordaro PD, Petronilho G. On the Borel property for solutions to systems of complex vector fields. Math. Nachr. 2013; 286(14-15):1439-1451.
2. Barostichi RF, Cordaro PD, Petronilho G. Strong unique continuation for systems of complex vector fields. Bull. Sci. Math. 2014; 138(4):457-469.
3. Bierstone E, Milman PD. Semianalytic and subanalytic sets. Inst. Hautes Etudes Sci. Publ. Math. 1988; 67:5-42.
4. Della Sala G, Cordaro PD, Lamel B. The Borel map in locally integrable structures. Mathematische Annalen, Feb 2019.
5. Della Sala G, Lamel B. Asymptotic approximations and a Borel-type result for CR functions. Internat. J. Math. 2013; 24(11):1350091.
6. Della Sala G, Lamel B. CR functions and their formal expansions. Bull. Inst. Math. Acad. Sin. (N.S.). 2013; 8(3):377-387.
7. Della Sala G, Lamel B. On the vanishing rate of smooth CR functions. Proc. Amer. Math. Soc. Ser. B. 2014; 1:23-32.
8. Gamelin TW. Uniform algebras. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.
9. Hardt RM. Some analytic bounds for subanalytic sets. In Differential geometric control theory (Houghton, Mich., 1982), volume 27 of Progr. Math., pages 259-267. Birkh¨auser Boston, Boston, MA, 1983.
10. Hironaka H. Subanalytic sets. In Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, pages 453–493. Kinokuniya, Tokyo, 1973.
11. K¨othe G. Topological vector spaces. II, volume 237 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]. Springer-Verlag, New York-Berlin, 1979.
12. Paulo D. Cordaro, Giuseppe Della Sala, Bernhard Lamel, The Borel map for compact sets in the plane, J. Funct. Anal, 2019, 1-12.