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Stability and Fixed Point Results on Partial b-Metric Space with regard to Quasi Contraction via (δ,Ü)–Convex Structure

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Abstract
In addition to introducing the idea of (δ,Ü)– Convex Partial *b*-Metric Spaces employing convex structure, this paper gives the stability theorems provided by the novel iteration strategy. Motivated by this method, we gave certain stability theorems and proved the existence and uniqueness of fixed points for T-mean non expanding maps. We show that our results considerably generalize previous fixed point results to the new idea of (δ,Ü)-Convex Partial b-Metric Spaces. In addition to improving knowledge of fixed point theory, this extension opens up new avenues for its use in increasingly intricate and varied mathematical contexts. As such, our study contributes to the area by providing a solid basis for further research and possible applications in a range of scientific and engineering fields.

Keywords: Stability, $(\delta, \ddot{U})$- Convex Structure**,** Convex Partial b-metric Space, Fixed Point

Introduction

Numerous generalisations of the Bannach contraction have been made in the literature. In 1968, Kannan [17] produced fixed point results for metric space for the mapping satisfying: $d(Tx,Ty) \leq \mu[d(x,Tx) + d(y,Ty)]$, for all $x, y \in H$ and $\mu \in (0,1)$. The concept of convexity was introduced by Takkahashi [25] in 1970, and a few fixed point solutions for non expanding mapping in metric space were found. By presenting a new contraction that, for all $x, y \in H$ and $a + b + c \leq 1$, where a, b, and c are nonnegative, obeyed the following formula: $d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty) + cd(x,y)$, Reich [22] extended and generalised the Bannach contraction finding in 1971. Bakhtin [2] and Czerwik [10] introduced the idea of b-metric space at separate times and demonstrated a number of fixed point theorems in b-metric spaces. Matthews [20] presented the concept of partial metric space and established a few fixed point theorems related to the Bannach contraction theorem in a distinct path of research. The concept of metric-like space, in which a point's self-distance need not be equal to zero, was first proposed by Amini-Harandi [15] in 2012. The author demonstrated a number of fixed point findings in partial metric and b-metric-like spaces within this innovative idea. Convex metric space was another generalisation. Later, fixed point results in convex metric space and convex b-metric spaces were examined by Beg and Abbas [3, 4, 5], Chang, Kim and Jin [5], Ciric [7], Ding [11], and several more scholars. Many scholars have since examined the partial metric space that results from the fixed point. Georgescu [13] investigated iterated function systems in the context of α-complate metric spaces in 2017 using partial generalised contractions. Fixed point results on orthogonal full metric spaces were established in 2020 by Sawangsup K. Sintunavarat W. and Cho Y.J. [23]. Chen [19] presented a fixed point theorem with strong convergence in 2021 for convex graphical rectangle b-metric space. The fixed point theorems for Reich contraction mapping in convex b-metric space that satisfy the Mann iteration sequence and weak T-stability were established by Karaca *et al.* in 2021 [18]. The concept of T-mean nonexpansive mapping was first presented by A.A. Mebawondu *et al*. in 2022. They demonstrated FP in metric space and provided other cases of T-mean nonexpansive mapping that did not occur on metric space. The new idea of (T,S)-stability in metric spaces was recently presented by Calderon *et al.* [6] in 2023, and they also produced some results on metric spaces.

Motivated by these groundbreaking findings, we investigate the existence of fixed points by presenting the concept of (δ, Ü)-convex partial b-metric space. We also establish stability theorems for iteration schemes in the context of partial b-metric space and prove fixed point results for various contractions for T-mean non expansive mappings. It is possible to expand these findings in different metric spaces.

1. Preliminaries:

We recall some basic definition which will be utilized in our paper:

Definition 2.1 [15]**:** Let H be a nonempty set the function $\Omega_p: H \times H \to [0,\infty]$ is called a partial metric on H if for all $u,v,w \in H$. Then the following conditions are satisfied:

(i) $0 \leq \Omega_p(u,u) \leq \Omega_p(u,v)$ (small self distance)

(ii) $\Omega_p(u,v) = \Omega_p(v,u)$ (symmetry)

(iii) $\Omega_p(u,u) = \Omega_p(u,v) = \Omega_c(v,v)$ iff $u = v$ (equality)

(iv) $\Omega_p(u,v) \leq \Omega_p(u,w) + \Omega_p(w,v) - \Omega_p(w,w)$ (triangularity)

Then the pair (H, Ω_p) is called partial metric spaces on H.

Definition 2.2 [15]**:** Let H be a nonempty set the function $\Omega_{pb}: H \times H \to [0,\infty]$ with $s \geq 1$ is called a partial *b*-metric on H if for all $u,v,w \in H$. Then the following conditions are satisfied:

(i) $0 \leq \Omega_{pb}(u,u) \leq \Omega_{pb}(u,v)$ (small self distance)

(ii) $\Omega_{pb}(u,v) = \Omega_{pb}(v,u)$ (symmetry)

(iii) $\Omega_{pb}(u,u) = \Omega_{pb}(u,v) = \Omega_{pb}(v,v)$ iff $u = v$ (equality)

(iv) $\Omega_{pb}(u,v) \leq s[\Omega_p(u,w) + \Omega_p(w,v) - \Omega_p(w,w)]$ (triangularity).

Then the pair (H, Ω_{pb}) is called partial *b*-metric spaces on H. the number s is called the coefficient of (H, Ω_{pb}).

Definition 2.3 [24]**:** Let a set $C \subseteq R$ and $\delta, \ddot{U} \in [0,1]$ then C is called $(\delta, \ddot{U})$*–convex* if

$$\alpha x\delta + (1-\alpha)\ddot{U}y \in C \text{ for all } x,y \in C \text{ and } \alpha \in [0,1].$$

Definition 2.3 [24]**:** Let a set $C \subseteq R$ and $\delta, \ddot{U} \in [0,1]$ then a mapping $F: C \subset R \to R$ is called $(\delta, \ddot{U})$*–convex* if C be a $(\delta, \ddot{U})$*–convex set and*

$$T(\alpha x\delta + (1-\alpha)\ddot{U}y) \leq \alpha\delta T(x) + (1-\alpha)\ddot{U}T(y) \text{ for all } x,y \in C \text{ and } \alpha \in [0,1].$$

Definition 2.4 [24]**:** Let H be a nonempty set and $I = [0,1]$. Define the function $\Omega_{cb}: H \times H \to [0,\infty]$ and a continuous function $\varpi: H \times H \times J \times I \to H$. then ϖ is called $(\delta, \ddot{U})$*–convex structure on* H *if:*

$$\Omega_{pb}(t, \varpi(\vartheta, \theta, \varphi; \alpha, \delta, \ddot{U})) \leq \alpha\delta\Omega_{pb}(t,\vartheta) + (1-\alpha)\ddot{U}\Omega_{pb}(t,\theta) \text{ for all } t \in H$$

$$\text{and } (\vartheta, \theta, \varphi; \alpha, \delta, \ddot{U}) \in H \times H \times J \times I, \quad \text{where } J \subseteq I.$$

Definition 2.5 [24]**:** Let the function $\varpi: H \times H \times J \times I \to H$ be a $(\delta, \ddot{U})$*–convex structure on a B-Partial metric space* (H, Ω_{pb}) and $I = [0,1]$ *then* (H, Ω_{pb}, ϖ) *is called* $(\delta, \ddot{U})$*–convex b-Partial metric space.*

Definition 2.6 [24]**:** (H, Ω_{pb}, ϖ) be a $(\delta, \ddot{U})$*– Convex b-Partial metric space with a* function $: H \to H$, for $\vartheta_k \in H$ and $\varsigma_k \in [0,1]$ then generalize Mann's iteration sequence (ϑ_k) is defined as:

$$\vartheta_{k+1} = \varpi(\vartheta_k, \Omega_{pb}\vartheta_k; \varsigma_k, \delta, \ddot{U}), \text{ for all } k \in N.$$

Definition 2.7 [15]**:** Let (H, Ω_{pb}) be a partial *b*-metric space with coefficient s and $\{\vartheta_k\}$ be a sequence in H then

(i) *The sequence* $\{\vartheta_n\}$ *is said to be convergent in* (H, Ω_{pb}) *and convergence to* $\vartheta^* \in H$, *if* $\lim_{n\to\infty} \Omega_{pb}(\vartheta_n, \vartheta^*) = \Omega_{pb}(\vartheta^*, \vartheta^*)$.

(ii) *The sequence* $\{\vartheta_n\}$ *is called Cauchy sequence in* (H, Ω_{pb}) *if* $\lim_{n,m\to\infty} \Omega_{pb}(\vartheta_n, \vartheta_m) =$ exists and finite.

(iii) (H, Ω_{pb}) *is said to be a complete if every Cauchy sequence is Convergent in H, there exists* ϑ^* *such that* $\lim_{n,m\to\infty} \Omega_{pb}(\vartheta_n, \vartheta_m) = \Omega_{pb}(\vartheta^*, \vartheta^*) = \lim_{n\to\infty} \Omega_{pb}(\vartheta_n, \vartheta^*)$.

Definition 2.8: Let (H, Ω_{pb}) be a partial *b*-metric space then a mapping $\Omega_{pb}: H \to H$ is said to be sequentially convergent if we have each sequence $\{\vartheta_n\}$ if $\{\Omega_{pb}(\vartheta_n)\}$ is convergent then $\{\vartheta_n\}$ is also convergent.

Definition 2.9: Let T, S be a self mappings on partial *b*-metric space and ϑ^* be the FP of S. Let $\{T(\vartheta_n)\}$ be a sequence generated by an iteration scheme, that is

$$T(\vartheta_{n+1}) = h(T, S, \vartheta_n), n \in N$$

Where ϑ_0 is initial point and h is function. Assume that a sequence $\{T(\vartheta_n)\}$ Converges to $\{T(\vartheta^*)\}$ and set $b_n = \Omega_{pb}(T(\vartheta_{n+1}), h(T, S, \vartheta_n)), n \in N$ then the iteration scheme, that is

$T(\vartheta_{n+1}) = h(T, S, \vartheta_n), n \in N$ *is called* (T, S) *–stable if and only if*

$\lim_{n\to\infty} b_n = 0 \Leftrightarrow \lim_{n\to\infty} T(\vartheta_n) = T(\vartheta_{n+1})$.

If we take $T = I$ then the definition 2.9 reduced to the concept of stability of an iteration scheme.

Lemma 1: Let T be a self mappings on partial *b*-metric space and continuous at $u \in H$ then each sequence $\{\vartheta_n\}$ such that $\vartheta_n \to u$, we have $T(\vartheta_n) \to T(u)$, that is $\lim_{n\to\infty} \Omega_{pb}(T(\vartheta_n), T(u)) = \Omega_{pb}(T(u), T(u))$.

Lemma 2: Let $\{\vartheta_n\}$ *and* $\{b_n\}$ be two non negative sequence and $0 \le \theta < 1$
Such that $\vartheta_{n+1} \le \theta\vartheta_n + b_n$, for all $n \in N$
If $\lim_{n\to\infty} b_n = 0$ *then* $\lim_{n\to\infty} \vartheta_n = 0$.
Now the following theorems prove by using (δ, Ü)-convex partial b-metric space

Our Main Results
Theorem 3.1: Let $(H_C, \Omega_{pb}, \varpi)$ be a complete $(\delta, \ddot{U})$*-convex Partial b-metric space* with constants $s \ge 1$ *and* $\Omega: H_C \to H_C$ be a continuous kannan type contraction mapping defined as:

$$\Omega_{pb}(\Omega\vartheta, \Omega\eta) \le \Lambda\,\{\Gamma(\vartheta, \Omega\vartheta) + \Gamma(\eta, \Omega\eta)\},\ for\ \vartheta, \eta \in H_C$$

and for some $\Lambda \in \left[0, \frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Omega_{pb}(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ *and*

$$define\ \vartheta_k = \varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U}) \text{ for } k \in \mathrm{N} \text{ and } 0 < \varsigma_{k-1} < \frac{1}{s}, and\ \delta, \ddot{U} \le [0,1).$$

Then Ω *has a unique FP*. Moreover $\{T(\vartheta_k)\}$ converges to $\{T(\vartheta^*)\}$, where ϑ^* *is FP.*

Proof: for $k \in \mathrm{N}$, we have

$$\Omega_{pb}(\vartheta_k, \vartheta_{k+1}) = \Omega_{pb}(\vartheta_k, \varpi(\vartheta_k, \Omega\vartheta_k; \varsigma_k, \delta, \ddot{U})) = \varsigma_k\,\delta\Omega_{pb}(\vartheta_k, \vartheta_k) + (1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k, \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \vartheta_{k+1}) \le (1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k, \Omega\vartheta_k).$$

Now,
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) = \Omega_{pb}(\varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U}), \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \varsigma_{k-1}\,\delta\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \varsigma_{k-1}\,s\delta[\Omega_{pb}(\vartheta_{k-1}, \vartheta_k) + \Omega_{pb}(\vartheta_k, \Omega\vartheta_k) - \Omega_{pb}(\Omega\vartheta_k, \Omega\vartheta_k)] + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \varsigma_{k-1}\,s\delta[\Omega_{pb}(\vartheta_{k-1}, \vartheta_k) + \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)] + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \varsigma_{k-1}\,s\delta(1-\varsigma_{k-1})\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \varsigma_{k-1}\,s\delta\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Lambda\{\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)\}$$
$$[1-\varsigma_{k-1}\,s\delta - (1-\varsigma_k)\ddot{U}\Lambda]\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le (1-\varsigma_{k-1})[\varsigma_{k-1}\,s\delta + \ddot{U}\Lambda]\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \frac{(1-\varsigma_{k-1})[\varsigma_{k-1}\,s\delta + \ddot{U}\Lambda]}{[1-\varsigma_{k-1}\,s\delta - (1-\varsigma_k)\ddot{U}\Lambda]}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$
Put $j = \frac{(1-\varsigma_{k-1})[\varsigma_{k-1}\,s\delta + \ddot{U}\Lambda]}{[1-\varsigma_{k-1}\,s\delta - (1-\varsigma_k)\ddot{U}\Lambda]} \le 1.$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le j\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

Since $\delta, \ddot{U} \in [0,1)$ *and* $j \le 1, \Lambda \in \left[0, \frac{1}{2}\right)$, $0 < \varsigma_{k-1} < \frac{1}{s}$, *and* $\Omega_{pb}(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ then gives $\Omega_{pb}(\vartheta_k, \Omega\vartheta_k)$ is non increasing sequence of positive reals. We follows $\{\vartheta_k\}$ is a Cauchy sequence. Now we shall prove ϑ^* is a fixed point of of Ω for this
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta^*) - \Omega_{pb}(\Omega\vartheta^*, \Omega\vartheta^*)]$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta^*)]$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s\Omega_{pb}(\vartheta^*, \vartheta_p) + s^2[\Gamma(\vartheta_p, \Omega\vartheta_p) + \Gamma(\Omega\vartheta_p, \Omega\vartheta^*)]$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s\Omega_{pb}(\vartheta^*, \vartheta_p) + s^2\Omega_{pb}(\vartheta_p, \Omega\vartheta_p) + s^2\Lambda[\Omega_{pb}(\vartheta_p, \Omega\vartheta_p) + \Omega_{pb}(\vartheta^*, \Omega\vartheta^*)]$$
$$[1-s^2\Lambda]\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s\Omega_{pb}(\vartheta^*, \vartheta_p) + (1+s^2)\Omega_{pb}(\vartheta_p, \Omega\vartheta_p)$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le \frac{s}{[1-s^2\Lambda]}\Omega_{pb}(\vartheta^*, \vartheta_p) + \frac{(1+s^2)}{[1-s^2\Lambda]}\Omega_{pb}(\vartheta_p, \Omega\vartheta_p)$$
As $p \to \infty$, *we get* $\lim_{p\to\infty} \Omega_{pb}(\vartheta^*, \Omega\vartheta^*) = 0$
So $\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) = 0 \Rightarrow \vartheta^* = \Omega\vartheta^*$

Hence ϑ^* is a fixed point of Ω.

Uniqueness: Suppose Θ is another fixed point of Ω then we have $\Omega\vartheta^* = \vartheta^*$ *and* $\Omega\Theta = \Theta$.
Now $\Omega_{pb}(\vartheta^*, \Theta) = \Omega_{pb}(\Omega\vartheta^*, \Omega\Theta)$

$$\le \Lambda\{\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) + \Omega_{pb}(\Theta, \Omega\Theta)\}$$
$$\le \Lambda\{\Omega_{pb}(\vartheta^*, \vartheta^*) + \Omega_{pb}(\Theta, \Theta)\}$$

$\Gamma(\vartheta^*, \Theta) = 0 \Rightarrow \vartheta^* = \Theta$

Hence proof.

Theorem 3.2: Under the hypotheses of theorem 3.1, the sequence $\Omega(\vartheta_n)$ is $(\Omega, S)-$ stable.

Proof: Suppose $\{\vartheta_k\}$ is an arbitrary sequence in H and $b_n = \Omega_{pb}\left(\Omega(\vartheta_{n+1}), \Omega(S(\vartheta_n))\right)$

Let $\lim_{n\to\infty} b_n = 0$ then

$$\Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) \le \Omega_{pb}\left(\Omega(\vartheta_{n+1}), \Omega(S(\vartheta_n))\right) + \Omega_{pb}(b_n, \Omega(\vartheta^*))$$
$$\Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) \le b_n + \frac{(1-\varsigma_{k-1})[\varsigma_{k-1} s\delta + \ddot{U}\Lambda]}{[1-\varsigma_{k-1} s\delta - (1-\varsigma_k)\ddot{U}\Lambda]}\Omega_{pb}\left(\Omega(S(\vartheta_n)), \Omega(S(\vartheta^*))\right)$$

But we have, $b_n = 0 \Leftrightarrow \lim_{n\to\infty} \Omega(\vartheta_n) = \Omega(\vartheta^*)$ *by lemma 2.*
Conversely, $\lim_{n\to\infty} \Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) = 0$ *then by help of lemma 2, we have*

$$b_n = \Omega_{pb}\left(\Omega(\vartheta_{n+1}), \Omega(S(\vartheta_n))\right)$$
$$\le \Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) + \Omega_{pb}\left(\Omega(\vartheta^*), \Omega(S(\vartheta_n))\right)$$
$$\le \Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) + \Omega_{pb}\left(\Omega(S(\vartheta_n)), \Omega(S(\vartheta^*))\right)$$
$$\le \Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) + \Lambda\{\Omega_{pb}(\vartheta_n, \Omega\vartheta_n) + \Omega_{pb}(\vartheta^*, \Omega\vartheta^*)\}$$
$$\le \Omega_{pb}(\Omega(\vartheta_{n+1}), \Omega(\vartheta^*)) + \Lambda\{\Omega_{pb}(\vartheta_n, \vartheta_n) + \Omega_{pb}(\vartheta^*, \vartheta^*)\}$$
$$b_n \to 0 \text{ as } n \to \infty$$

Therefore $\Omega(\vartheta_n)$ is $(\Omega, S)-$ stable.

Theorem 3.3: Let $(H_C, \Omega_{pb}, \varpi)$ be a complete $(\delta, \ddot{U})$*-convex Partial b-metric space* with constants $and\ \Omega: H_C \to H_C$ be a self mapping defined as:

$$\Omega_{pb}(\Omega\vartheta, \Omega\eta) \le \Lambda\, Max\{\Omega_{pb}(\vartheta, \eta), \Omega_{pb}(\vartheta, \Omega\vartheta), \Omega_{pb}(\eta, \Omega\eta)\},\ for\ \vartheta, \eta \in H_C$$

and for some $\Lambda \in \left[0, \frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Omega_{pb}(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ *and define* $\vartheta_k = \varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U})$ for $k \in \mathrm{N}$ and $0 < \varsigma_{k-1} < \frac{1}{s^2}$, $\delta, \ddot{U} \le [0,1)$.
Then Ω *has a unique fixed point of* H_C.

Proof: for $k \in \mathrm{N}$, we have

$$\Omega_{pb}(\vartheta_k, \vartheta_{k+1}) = \Omega_{pb}(\vartheta_k, \varpi(\vartheta_k, \Omega\vartheta_k; \varsigma_k, \delta, \ddot{U})) = \varsigma_k \delta\Omega_{pb}(\vartheta_k, \vartheta_k) + (1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k, \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \vartheta_{k+1}) \le (1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k, \Omega\vartheta_k).$$

Now,

$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) = \Omega_{pb}(\varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \ddot{U}), \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \varsigma_{k-1}\delta\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \le \varsigma_{k-1} s\delta\left[\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k) - \Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_{k-1})\right] + (1-\varsigma_k)\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$
$$\le s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Omega_{pb}(\Omega\vartheta_{k-1}, \Omega\vartheta_k)$$
$$\le s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$
$$+\{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\{\Omega_{pb}(\vartheta_{k-1}, \vartheta_k), \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)\}$$
$$\le s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$
$$+\{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\left\{\begin{matrix}(1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)\end{matrix}\right\}$$

Where $\beta = (1-\varsigma_{k-1})\ddot{U} \le 1$

$$\le s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$
$$+\{s\delta\varsigma_{k-1} + \beta\}\Lambda\, Max\left\{\begin{matrix}\beta\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)\end{matrix}\right\}$$

There are two possibilities:

Case (i) if $Max\left\{\begin{matrix}\beta\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)\end{matrix}\right\} = \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$
Then

$$\le s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\,\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$$

$\leq \{s\delta\varsigma_{k-1} + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\} \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Put $K_1 = \{s\delta\varsigma_{k-1} + \{s\delta\varsigma_{k-1} + \beta\}\Lambda\} \leq 1$

$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \leq K_1 \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Since $0 < \varsigma_{k-1} < \frac{1}{2}, \quad \delta, \text{Ü} \leq 1, \Lambda \in \left[0, \frac{1}{2}\right)$ $then\ K_1 \leq 1.$

Case (ii) if $Max\left\{\begin{matrix} \beta\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}), \Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \end{matrix}\right\} = \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)$

Then

$\leq s\delta\varsigma_{k-1} \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + \beta\}\Lambda \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)$

$[1 - \{s\delta\varsigma_{k-1} + \beta\}\Lambda]\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \leq s\delta\varsigma_{k-1} \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \leq \frac{s\delta\varsigma_{k-1}}{[1 - \{s\delta\varsigma_{k-1} + \beta\}\Lambda]} \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Put $K_2 = \frac{s\delta\varsigma_{k-1}}{[1 - \{s\delta\varsigma_{k-1} + \beta\}\Lambda]} \leq 1$

$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \leq K_2 \Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Since $0 < \varsigma_{k-1} < \frac{1}{2}, \quad \delta, \text{Ü} \leq 1, \Lambda \in \left[0, \frac{1}{2}\right)$ $then\ K_1 \leq 1.$

Let $K = Max\{K_1, K_2\}$ then

$\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) \leq K\Omega_{pb}(\vartheta_{k-1}, \Omega\vartheta_{k-1})$

Since $\delta, \text{Ü}, \varsigma_{k-1} \in \left[0, \frac{1}{2}\right)$ $and\ K \leq 1, \Omega_{pb}(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ then applying $k \to \infty$ then we get $\lim_{k\to\infty} \Omega_{pb}(\vartheta_k, \Omega\vartheta_k) = 0$ and since, $\Gamma(\vartheta_k, \vartheta_{k+1}) \leq (1 - \varsigma_k)\text{Ü}\Omega_{pb}(\vartheta_k, \Omega\vartheta_k) < \Omega_{pb}(\vartheta_k, \Omega\vartheta_k)$

Thus, $\lim_{k\to\infty} \Omega_{pb}(\vartheta_k, \vartheta_{k+1}) = 0$

Now we show that $\{\vartheta_k\}$ is a Cauchy sequence. Suppose $\{\vartheta_k\}$ is not a Cauchy sequence and let two subsequences $\{\vartheta_{p_t}\}$ and $\{\vartheta_{q_t}\}$ of $\{\vartheta_k\}$ such that $q_t > p_t > t$ satisfying $\Gamma\left(\vartheta_{p_t}, \vartheta_{q_t}\right) \geq \epsilon\ and\ \Gamma\left(\vartheta_{p_t-1}, \vartheta_{q_t}\right) < \epsilon$ then by $triangularity$ property

$$\epsilon \leq \Omega_{pb}\left(\vartheta_{p_t}, \vartheta_{q_t}\right) \leq s\left[\Omega_{pb}\left(\vartheta_{p_t}, \vartheta_{q_t+1}\right) + \Omega_{pb}\left(\vartheta_{q_t+1}, \vartheta_{q_t}\right) - \Omega_{pb}\left(\vartheta_{q_t+1}, \vartheta_{q_{t+1}}\right)\right]$$

Taking $t \to \infty$ and applying above values then we get

$$\lim_{t\to\infty} \sup \Omega_{pb}\left(\vartheta_{p_t}, \vartheta_{q_t}\right) \leq \epsilon$$

Now, $\Omega_{pb}\left(\vartheta_{p_t}, \vartheta_{q_t+1}\right) = \Omega_{pb}\left(\left(\varpi(\vartheta_{p_t-1}, \Omega\vartheta_{p_t-1}; \varsigma_{p_t-1}, \delta, \text{Ü}), \vartheta_{p_t+1}\right)\right)$

$\leq \delta\varsigma_{p_t-1} \Omega_{pb}\left(\vartheta_{p_t-1}, \vartheta_{p_t+1}\right) + (1 - \varsigma_{p_t-1})\text{Ü}\Omega_{pb}\left(\Omega\vartheta_{p_t-1}, \vartheta_{p_t+1}\right)$

$\leq \delta\varsigma_{p_t-1} \Omega_{pb}\left(\vartheta_{p_t-1}, \vartheta_{p_t+1}\right)$

$+(1 - \varsigma_{p_t-1})\text{Ü}s[\Omega_{pb}(\Omega\vartheta_{p_t-1}, \vartheta_{p_t}) + \Omega_{pb}(\vartheta_{p_t}, \vartheta_{p_t+1})]$

$\{1 - (1 - \varsigma_{p_t-1})\text{Ü}s\}\Omega_{pb}(\vartheta_{p_t}, \vartheta_{p_t+1}) \leq \delta\varsigma_{p_t-1} \Omega_{pb}\left(\vartheta_{p_t-1}, \vartheta_{p_t+1}\right)$

$+(1 - \varsigma_{p_t-1})\text{Ü}s\ \Omega_{pb}(\Omega\vartheta_{p_t-1}, \vartheta_{p_t})$

Therefore letting $t \to \infty$ then we obtain $\lim_{t\to\infty} Sup\ \Omega_{pb}(\vartheta_{p_t}, \vartheta_{p_t+1}) \leq \epsilon,$ which is contradiction. Thus $\{\vartheta_k\}$ is a Cauchy sequence H_C. By completeness of H_C there exists $\vartheta^* \in H_C$ such that $\lim_{t\to\infty} \Omega_{pb}(\vartheta_p, \vartheta^*) = 0.$

Now we claim ϑ^* is a fixed point of $\Omega.$

$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \leq s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta^*) - \Omega_{pb}(\Omega\vartheta^*, \Omega\vartheta^*)]$

$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \leq s^2[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta_p)] + s[\Omega_{pb}(\Omega\vartheta_p, \Omega\vartheta^*)]$

$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \leq s^2[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta_p)] + \Lambda sMax\{\Omega_{pb}(\vartheta_p, \vartheta^*), \Omega_{pb}(\vartheta_p, \Omega\vartheta_p), \Omega_{pb}(\vartheta^*, \Omega\vartheta^*)\}$

$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \leq s^2[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta_p)] + \Lambda sMax\{\Omega_{pb}(\vartheta_p, \vartheta^*), \Omega_{pb}(\vartheta_p, \Omega\vartheta_p), \Omega_{pb}(\vartheta^*, \Omega\vartheta^*)\}$

Letting $t \to \infty$ $then\ we\ get$

$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \leq \Lambda sMax\{\Omega_{pb}(\vartheta^*, \Omega\vartheta^*), s\Omega_{pb}(\vartheta^*, \Omega\vartheta^*)\}$

$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \leq \Lambda s^2\Omega_{pb}(\vartheta^*, \Omega\vartheta^*)$

So $\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) = 0 \Rightarrow \vartheta^* = \Omega\vartheta^*$

Hence ϑ^* is a fixed point of $\Omega.$

Uniqueness: Suppose Θ is another fixed point of Ω then we have $\Omega\vartheta^* = \vartheta^*$ and $\Omega\Theta = \Theta$.

Now $\Omega_{pb}(\vartheta^*,\Theta) = \Omega_{pb}(\Omega\vartheta^*,\Omega\Theta)$

$$\leq \Lambda Max\{\Omega_{pb}(\vartheta^*,\Theta),\Omega_{pb}(\vartheta^*,\Omega\vartheta^*),\Omega_{pb}(\Theta,\Omega\Theta)\}$$
$$\leq \Lambda Max\{\Omega_{pb}(\vartheta^*,\Theta),\Omega_{pb}(\vartheta^*,\vartheta^*),\Omega_{pb}(\Theta,\Theta)\}$$
$$\Omega_{pb}(\vartheta^*,\Theta) \leq \Lambda\Omega_{pb}(\vartheta^*,\Theta)$$
$$(1-\Lambda)\ \Omega_{pb}(\vartheta^*,\Theta) \leq 0$$
$$(1-\Lambda) \neq 0,\ \Omega_{pb}(\vartheta^*,\Theta) = 0 \Rightarrow \vartheta^* = \Theta$$

Hence proof.

Now by using definition 2.9: We achieve the following stability result.

Theorem 3.4: Under the hypotheses of theorem 3.1, the sequence $\Omega(\vartheta_n)$ is $(\Omega,S)-$ stable.

Proof: Suppose $\{\vartheta_k\}$ is an arbitrary sequence in H and $b_n = \Omega_{pb}\left(\Omega(\vartheta_{n+1}),\Omega(S(\vartheta_n))\right)$

Let $\lim_{n\to\infty} b_n = 0$ then

$$\Omega_{pb}(\Omega(\vartheta_{n+1}),\Omega(\vartheta^*)) \leq \Omega_{pb}\left(\Omega(\vartheta_{n+1}),\Omega(S(\vartheta_n))\right) + \Omega_{pb}(b_n,\Omega(\vartheta^*))$$
$$\Omega_{pb}(\Omega(\vartheta_{n+1}),\Omega(\vartheta^*)) \leq b_n + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega(S(\vartheta_n)),\Omega(S(\vartheta^*)))$$

But we have, $b_n = 0 \Leftrightarrow \lim_{n\to\infty}\Omega(\vartheta_n) = \Omega(\vartheta^*)$ *by lemma 2.*

Conversely, $\lim_{n\to\infty}\Omega_{pb}(\Omega(\vartheta_{n+1}),\Omega(\vartheta^*)) = 0$ *then by help of lemma 2, we have*

$$b_n = \Omega_{pb}\left(\Omega(\vartheta_{n+1}),\Omega(S(\vartheta_n))\right)$$
$$\leq \Omega_{pb}(\Omega(\vartheta_{n+1}),\Omega(\vartheta^*)) + \Omega_{pb}\left(\Omega(\vartheta^*),\Omega(S(\vartheta_n))\right)$$
$$\leq \Omega_{pb}(\Omega(\vartheta_{n+1}),\Omega(\vartheta^*)) + \Omega_{pb}\left(\Omega(S(\vartheta_n)),\Omega(S(\vartheta^*))\right)$$
$$\leq \Omega_{pb}(\Omega(\vartheta_{n+1}),\Omega(\vartheta^*)) + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega(\vartheta_n),\Omega(\vartheta^*))$$
$$b_n \to 0 \ as\ n\to\infty$$

Therefore $\Omega(\vartheta_n)$ is $(\Omega,S)-$ stable.

Theorem 3.5: Let (H_C,Ω_{pb},ϖ) be a complete $(\delta,\ddot{U})$-*convex Partial b-metric space* with constants $s \geq 1$ and $\Omega: H_C \to H_C$ be a quasi-contraction mapping defined as:

$$\Omega_{pb}(\Omega\vartheta,\Omega\eta) \leq \Lambda\ Max\{\Omega_{pb}(\vartheta,\eta),\Omega_{pb}(\vartheta,\Omega\vartheta),\Omega_{pb}(\eta,\Omega\eta),\Omega_{pb}(\vartheta,\Omega\eta),\Omega_{pb}(\eta,\Omega\vartheta)\},\ for\ \vartheta,\eta \in H_C$$

and for some $\Lambda \in \left[0,\frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Omega_{pb}(\vartheta_0,\Omega\vartheta_0) = \mathfrak{M} < \infty$ and

$define\ \vartheta_k = \varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U})$ for $k \in \mathrm{N}$ and $0 < \varsigma_{k-1} < \frac{1}{s^2},\delta,\ddot{U} \leq [0,1)$. Then Ω *has a unique FP.*

Proof: for $k \in \mathrm{N}$, we have

$$\Omega_{pb}(\vartheta_k,\vartheta_{k+1}) = \Omega_{pb}\left(\vartheta_k,\varpi(\vartheta_k,\Omega\vartheta_k;\varsigma_k,\delta,\ddot{U})\right) = \varsigma_k\ \delta\Omega_{pb}(\vartheta_k,\vartheta_k) + (1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k,\vartheta_{k+1}) \leq (1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k,\Omega\vartheta_k).$$

Now,

$$\Omega_{pb}(\vartheta_k,\Omega\vartheta_k) = \Omega_{pb}(\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U}),\Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k,\Omega\vartheta_k) \leq \varsigma_{k-1}\ \delta\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_k) + (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1},\Omega\vartheta_k)$$
$$\Omega_{pb}(\vartheta_k,\Omega\vartheta_k) \leq \varsigma_{k-1}\ s\delta\left[\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \Omega_{pb}(\Omega\vartheta_{k-1},\Omega\vartheta_k) - \Omega_{pb}(\Omega\vartheta_{k-1},\Omega\vartheta_{k-1})\right] + (1-\varsigma_k)\ddot{U}\Omega_{pb}(\Omega\vartheta_{k-1},\Omega\vartheta_k)$$
$$\leq s\delta\varsigma_{k-1}\ \Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}) + \{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Omega_{pb}(\Omega\vartheta_{k-1},\Omega\vartheta_k)$$
$$\leq s\delta\varsigma_{k-1}\ \Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$
$$+\{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\ Max\begin{Bmatrix} \Omega_{pb}(\vartheta_{k-1},\vartheta_k),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_k,\Omega\vartheta_k),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_k),\Omega_{pb}(\vartheta_k,\Omega\vartheta_{k-1}) \end{Bmatrix}$$
$$\leq s\delta\varsigma_{k-1}\ \Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$
$$+\{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\ Max\begin{Bmatrix} (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_k,\Omega\vartheta_k),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_k),\Omega_{pb}(\vartheta_k,\Omega\vartheta_{k-1}) \end{Bmatrix}$$
$$\leq s\delta\varsigma_{k-1}\ \Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$
$$+\{s\delta\varsigma_{k-1} + (1-\varsigma_{k-1})\ddot{U}\}\Lambda\ Max\begin{Bmatrix} (1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}), \\ \Omega_{pb}(\vartheta_k,\Omega\vartheta_k),\{\Omega_{pb}(\vartheta_{k-1},\vartheta_k) + \Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\}, \\ \Omega_{pb}(\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U}),\Omega\vartheta_{k-1}) \end{Bmatrix}$$
$$\leq s\delta\varsigma_{k-1}\ \Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

$$+\{s\delta\varsigma_{k-1}+(1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\left\{\begin{array}{c}(1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Omega_{pb}(\vartheta_k,\Omega\vartheta_k),\{(1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\},\\ \Omega_{pb}(\varpi(\vartheta_{k-1},\Omega\vartheta_{k-1};\varsigma_{k-1},\delta,\ddot{U}),\Omega\vartheta_{k-1})\end{array}\right\}$$

$$\leq s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

$$+\{s\delta\varsigma_{k-1}+(1-\varsigma_{k-1})\ddot{U}\}\Lambda\, Max\left\{\begin{array}{c}(1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \Omega_{pb}(\vartheta_k,\Omega\vartheta_k),\{(1-\varsigma_{k-1})\ddot{U}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\},\\ \varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+(1-\varsigma_{k-1})\Omega_{pb}(\Omega\vartheta_{k-1},\Omega\vartheta_{k-1})\end{array}\right\}$$

$$\leq s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

$$+\{s\delta\varsigma_{k-1}+\xi\}\Lambda\, Max\left\{\begin{array}{c}\xi\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\Omega_{pb}(\vartheta_k,\Omega\vartheta_k),\\ \{\xi\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\},\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})\end{array}\right\}$$

Where $\xi = (1-\varsigma_{k-1})\ddot{U} \leq 1$

$$\Gamma(\vartheta_k,\Omega\vartheta_k)\leq s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\{s\delta\varsigma_{k-1}+\xi\}\Lambda\, Max\left\{\begin{array}{c}\xi\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1}),\\ \{\xi\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\}\end{array}\right\}$$

$$\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\leq s\delta\varsigma_{k-1}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\{s\delta\varsigma_{k-1}+\xi\}\Lambda\,\xi\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})+\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)$$

$$[1-\{s\delta\varsigma_{k-1}+\xi\}\Lambda]\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\leq[s\delta\varsigma_{k-1}+\{s\delta\varsigma_{k-1}+\xi\}\Lambda\xi]\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

$$\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)\leq\frac{[s\delta\varsigma_{k-1}+\{s\delta\varsigma_{k-1}+\xi\}\Lambda\xi]}{[1-\{s\delta\varsigma_{k-1}+\xi\}\Lambda]}\Omega_{pb}(\vartheta_{k-1},\Omega\vartheta_{k-1})$$

Since $\delta,\ddot{U}\in[0,1),\Lambda\in\left[0,\frac{1}{2}\right),\quad 0<\varsigma_{k-1}<\frac{1}{s^2}, and\quad \xi\leq 1,\Omega_{pb}(\vartheta_0,\Omega\vartheta_0)=\mathfrak{M}<\infty$. then applying $k\to\infty$ then we get $\lim_{k\to\infty}\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)=0$ and since, $\Omega_{pb}(\vartheta_k,\vartheta_{k+1})\leq(1-\varsigma_k)\ddot{U}\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)<\Omega_{pb}(\vartheta_k,\Omega\vartheta_k)$

Thus, $\lim_{k\to\infty}\Omega_{pb}(\vartheta_k,\vartheta_{k+1})=0$

Now we show that $\{\vartheta_k\}$ is a Cauchy sequence. Suppose $\{\vartheta_k\}$ is not a Cauchy sequence and let two subsequences $\{\vartheta_{p_t}\}$ and $\{\vartheta_{q_t}\}$ of $\{\vartheta_k\}$ such that $q_t>p_t>t$ satisfying $\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{q_t}\right)\geq\epsilon\ and\ \Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{q_t}\right)<\epsilon$ then by $triangularity$ property

$$\epsilon\leq\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{q_t}\right)\leq s\left[\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{q_t+1}\right)+\Omega_{pb}\left(\vartheta_{q_t+1},\vartheta_{q_t}\right)-\Omega_{pb}\left(\vartheta_{q_t+1},\vartheta_{q_{t+1}}\right)\right]$$

Taking $t\to\infty$ and applying above values then we get

$$\limsup_{t\to\infty}\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{q_t}\right)\leq\epsilon$$

Now, $\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{q_t+1}\right)=\Omega_{pb}\left((\varpi(\vartheta_{p_t-1},\Omega\vartheta_{p_t-1};\varsigma_{p_t-1},\delta,\ddot{U}),\vartheta_{p_t+1})\right)$

$$\leq\delta\varsigma_{p_t-1}\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t+1}\right)+(1-\varsigma_{p_t-1})\ddot{U}\Omega_{pb}\left(\Omega\vartheta_{p_t-1},\vartheta_{p_t+1}\right)$$

$$\leq\delta\varsigma_{p_t-1}\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t+1}\right)$$

$$+(1-\varsigma_{p_t-1})\ddot{U}s[\Omega_{pb}(\Omega\vartheta_{p_t-1},\Omega\vartheta_{p_t+1})+\Omega_{pb}(\Omega\vartheta_{p_t+1},\vartheta_{p_t+1})]$$

$$\leq\delta\varsigma_{p_t-1}\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t+1}\right)+(1-\varsigma_{p_t-1})\ddot{U}s\Omega_{pb}(\Omega\vartheta_{p_t+1},\vartheta_{p_t+1})$$

$$+(1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\, Max\left\{\begin{array}{c}\Omega_{pb}(\vartheta_{p_t-1},\vartheta_{p_t+1}),\Omega_{pb}(\vartheta_{p_t-1},\Omega\vartheta_{p_t-1}),\\ \Omega_{pb}(\vartheta_{p_t+1},\Omega\vartheta_{p_t+1}),\Omega_{pb}(\vartheta_{p_t-1},\Omega\vartheta_{p_t+1}),\\ \Omega_{pb}(\vartheta_{p_t+1},\Omega\vartheta_{p_t-1})\end{array}\right\}$$

$$\leq\delta\varsigma_{p_t-1}s\left[\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t}\right)+\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{p_t+1}\right)\right]+(1-\varsigma_{p_t-1})\ddot{U}s\Omega_{pb}(\Omega\vartheta_{p_t+1},\vartheta_{p_t+1})$$

$$+(1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\, Max\left\{\begin{array}{c}s\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t}\right)+s\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{p_t+1}\right),\\ \Omega_{pb}(\vartheta_{p_t-1},\Omega\vartheta_{p_t-1}),\Omega_{pb}(\vartheta_{p_t+1},\Omega\vartheta_{p_t+1}),\\ s\Omega_{pb}(\vartheta_{p_t-1},\vartheta_{p_t+1})+s\Omega_{pb}(\vartheta_{p_t+1},\Omega\vartheta_{p_t+1}),\\ s\Omega_{pb}(\vartheta_{p_t+1},\vartheta_{p_t-1})+s\Omega_{pb}(\vartheta_{p_t-1},\Omega\vartheta_{p_t-1})\end{array}\right\}$$

$$\leq\delta\varsigma_{p_t-1}s\left[\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t}\right)+\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{p_t+1}\right)\right]+(1-\varsigma_{p_t-1})\ddot{U}s\Omega_{pb}(\Omega\vartheta_{p_t+1},\vartheta_{p_t+1})$$

$$+(1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\, Max\left\{\begin{array}{c}s\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t}\right)+s\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{p_t+1}\right),\\ \Omega_{pb}(\vartheta_{p_t-1},\Omega\vartheta_{p_t-1}),\Omega_{pb}(\vartheta_{p_t+1},\Omega\vartheta_{p_t+1}),\\ s\left\{s\left[\Omega_{pb}\left(\vartheta_{p_t-1},\vartheta_{p_t}\right)+\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{p_t+1}\right)\right]\right\}+s\Omega_{pb}(\vartheta_{p_t+1},\Omega\vartheta_{p_t+1}),\\ s\left\{s\left[\Omega_{pb}\left(\vartheta_{p_t+1},\vartheta_{p_t}\right)+\Omega_{pb}\left(\vartheta_{p_t},\vartheta_{p_t-1}\right)\right]\right\}+s\Omega_{pb}(\vartheta_{p_t-1},\Omega\vartheta_{p_t-1})\end{array}\right\}$$

$$\leq\delta\varsigma_{p_t-1}s\epsilon+(1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda\, Max\{s\epsilon,s^2\epsilon\}$$

$$\leq\delta\varsigma_{p_t-1}s\epsilon+(1-\varsigma_{p_t-1})\ddot{U}s\,\Lambda s^2\epsilon$$

Letting $t\to\infty$ then we get $\limsup_{t\to\infty}\Gamma\left(\vartheta_{p_t},\vartheta_{q_t+1}\right)<\epsilon$. This is contradiction. Thus $\{\vartheta_{p_t}\}$ being Cauchy sequence in H. By completeness of H, there exist $\vartheta^*\in H$ such that $\lim_{t\to\infty}\Omega_{pb}(\vartheta_p,\vartheta^*)=0$.

Now we shall prove ϑ^* is a fixed point of of Ω for this

$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta^*) - \Omega_{pb}(\Omega\vartheta^*, \Omega\vartheta^*)]$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta_p)] + s^2[\Omega_{pb}(\Omega\vartheta_p, \Omega\vartheta^*)]$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta_p)] + \Lambda s^2 Max\{\Omega_{pb}(\vartheta_p, \vartheta^*), \Omega_{pb}(\vartheta_p, \Omega\vartheta_p), \Omega_{pb}(\vartheta^*, \Omega\vartheta^*), \Omega_{pb}(\vartheta_p, \Omega\vartheta^*), \Omega_{pb}(\vartheta^*, \Omega\vartheta_p)\}$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le s[\Omega_{pb}(\vartheta^*, \vartheta_p) + \Omega_{pb}(\vartheta_p, \Omega\vartheta_p)] + \Lambda s^2 Max\left\{\begin{matrix}\Omega_{pb}(\vartheta_p, \vartheta^*), \Omega_{pb}(\vartheta_p, \Omega\vartheta_p), \Omega_{pb}(\vartheta^*, \Omega\vartheta^*), \\ s\Omega_{pb}(\vartheta_p, \vartheta^*) + s\Omega_{pb}(\vartheta^*, \Omega\vartheta^*), s\Omega_{pb}(\vartheta^*, \vartheta_p) + s\Omega_{pb}(\vartheta_p, \Omega\vartheta_p)\end{matrix}\right\}$$

Letting $t \to \infty$ *then we get*

$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le \Lambda s^2 Max\{\Omega_{pb}(\vartheta^*, \Omega\vartheta^*), s\Omega_{pb}(\vartheta^*, \Omega\vartheta^*)\}$$
$$\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) \le \Lambda s^2 \Omega_{pb}(\vartheta^*, \Omega\vartheta^*)$$

So $\Omega_{pb}(\vartheta^*, \Omega\vartheta^*) = 0 \Rightarrow \vartheta^* = \Omega\vartheta^*$

Hence ϑ^* is a fixed point of of Ω.

Uniqueness: Suppose Θ is another fixed point of Ω then we have $\Omega\vartheta^* = \vartheta^*$ *and* $\Omega\Theta = \Theta$.

Now $\Omega_{pb}(\vartheta^*, \Theta) = \Omega_{pb}(\Omega\vartheta^*, \Omega\Theta)$

$$\le \Lambda Max\{\Omega_{pb}(\vartheta^*, \Theta), \Omega_{pb}(\vartheta^*, \Omega\vartheta^*), \Omega_{pb}(\Theta, \Omega\Theta), \Omega_{pb}(\vartheta^*, \Omega\Theta), \Omega_{pb}(\Theta, \Omega\vartheta^*)\}$$
$$\le \Lambda Max\{\Omega_{pb}(\vartheta^*, \Theta), \Omega_{pb}(\vartheta^*, \vartheta^*), \Omega_{pb}(\Theta, \Theta), \Omega_{pb}(\vartheta^*, \Theta), \Omega_{pb}(\Theta, \vartheta^*)\}$$
$$\Omega_{pb}(\vartheta^*, \Theta) \le \Lambda \Omega_{pb}(\vartheta^*, \Theta)$$
$$(1-\Lambda)\, \Omega_{pb}(\vartheta^*, \Theta) \le 0$$
$$(1-\Lambda) \ne 0, \Omega_{pb}(\vartheta^*, \Theta) = 0 \Rightarrow \vartheta^* = \Theta$$

Hence proof.

Theorem 3.6: Let $(H_C, \Omega_{pb}, \varpi)$ be a complete $(\delta, \text{Ü})$*-convex Partial b-metric space* with constants $s \ge 1$ *and* $\Omega: H_C \to H_C$ be a Reich type contraction mapping defined as:

$$\Omega_{pb}(\Omega\vartheta, \Omega\eta) \le \theta\Omega_{pb}(\vartheta, \eta) + \Lambda\{\Omega_{pb}(\vartheta, \Omega\vartheta) + \Omega_{pb}(\eta, \Omega\eta)\}, \ for\ \vartheta, \eta \in H_C$$

and for some $\Lambda \in \left[0, \frac{1}{2}\right)$. Take $\vartheta_0 \in H_C$ such that $\Omega_{pb}(\vartheta_0, \Omega\vartheta_0) = \mathfrak{M} < \infty$ *and*

$define\ \vartheta_k = \varpi(\vartheta_{k-1}, \Omega\vartheta_{k-1}; \varsigma_{k-1}, \delta, \text{Ü})$ for $k \in \mathrm{N}$ and $0 < \varsigma_{k-1} < \frac{1}{s}$, and $\delta, \text{Ü} \le [0,1)$. Then Ω *has a unique FP.*

Proof: similar prove as Theorem **3.1.**

Conclusion

In this study, we introduced the idea of (δ,Ü)-convex structure and proved fixed point results for various contraction types in relation to partial b-metric spaces. We also showed that type T-mean non expansive mappings can be studied in various spaces, including hyperbolic metric spaces, and we investigated the existence of fixed points and provided stability with T-iteration schemes.

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