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Common Fixed Point Theorem in Controlled Metric Spaces

[1] **Shivani Chourasiya,** [2] **Kavita Shrivastava**
[1, 2] Department of Mathematics and Statistics, Dr. Harisingh Gour Vishwavidyalaya, Sagar M.P, India
(shivanichourasiya10@gmail.com, kavita.rohit@rediff.com)

DOI: https://doi.org/10.62225/2583049X.2024.4.5.3225 Corresponding Author: **Shivani Chourasiya**

Abstract
In the current paper, we derive common fixed-point theorem for generalized rational contractions that involve specific control functions in a controlled metric space. Numerous renowned findings from the literature are modified by our findings.

Keywords: Fixed Point, Rational Contraction, Controlled Metric Spaces

1. Introduction and Preliminaries

In 1906, Fréchet established the concept of a metric space. In 1922 Banach [1] gave contraction mapping principle. Many researchers generalized and extended this principle in different spaces like semi-metric, pseudo-metric, extended b-metric, partial, b-metric, and controlled metric spaces [see, 6-10] Czerwik [2] gave the notion of b-metric space as follows:

Definition 1.1. [2] Let $\Delta \neq \phi$ and $\Omega: \Delta \times \Delta \rightarrow [0, \infty)$. If,
1. $\Omega(\mu, \nu) = 0 \Leftrightarrow \mu = \nu$;
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$ for all $\mu, \nu \in \Delta$;
3. $\Omega(\mu, w) \leqslant s[\Omega(\mu, \nu) + \Omega(\nu, w)]$ for all $\mu, \nu, w \in \Delta$

Then (Δ, Ω) is said to be a b-metric space.
In 2017 Kamran *et al.* [3] defined the notion of a extended b-metric space as follows:

Definition 1.2. [3] Let $\Delta \neq \phi$ and $\sigma: \Delta \times \Delta \rightarrow [1, \infty)$ and $\Omega: \Delta \times \Delta \rightarrow [0, \infty)$. If,
1. $\Omega(\mu, \nu) = 0 \Leftrightarrow \mu = \nu$;
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$ for all $\mu, \nu \in \Delta$;
3. $\Omega(\mu, \nu) \leqslant \sigma(\mu, \nu)[\Omega(\mu, w) + \Omega(\nu, w)]$ for all $\mu, \nu, w \in \Delta$

Then (Δ, Ω) is said to be an extended b-metric spaces.
In 2018, Mlaiki *et al.* [5] gave a new type of extended b-metric space called controlled metric space as follows:

Definition 1.3. [5] Let $\Delta \neq \phi$ and $\sigma: \Delta \times \Delta \rightarrow [1, \infty)$ and $\Omega: \Delta \times \Delta \rightarrow [0, \infty)$. If,
1. $\Omega(\mu, \nu) = 0 \Leftrightarrow \mu = \nu$;
2. $\Omega(\mu, \nu) = \Omega(\nu, \mu)$ for all $\mu, \nu \in \Delta$;
3. $\Omega(\mu, \nu) \leqslant \sigma(\mu, w)\Omega(\mu, w) + \sigma(\nu, w)\Omega(\nu, w)$ for *all* $\mu, \nu, w \in \Delta$

Then (Δ, Ω, σ) is said to be a controlled metric space.

Definition 1.4. [5] Let $\{\mu_n\}_{n\geq 0}$ be a sequence in (Δ, Ω, σ).
1. $\{\mu_n\}_{n\geq 0} \rightarrow 0$ in Δ, is convergent if $\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbf{N}$ such that $\Omega(\mu_n, \mu) < \epsilon, \forall n \geq \mathbf{N}$.
2. $\{\mu_n\}_{n\geq 0}$ is Cauchy, if $\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbf{N}$ such that $\Omega(\mu_m, \mu_n) < \epsilon, \forall m, n \geq \mathbf{N}$.
3. If every Cauchy sequence in (Δ, Ω, σ) is convergent then (Δ, Ω, σ) is complete.

Theorem 1.1. [5] Let (Δ, Ω, σ) be a complete controlled metric space. Let $T: \Delta \rightarrow \Delta$ be such that

$$\Omega(T\mu, T\nu) \leq \gamma\, \Omega(\mu, \nu), \forall \mu, \nu \in \Omega$$

Where $\gamma \in [0, 1)$. For $\mu_0 \in \Delta$, take $\mu_n = T^n\mu_0$. Suppose that,

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\mu_{i+1},\mu_{i+2})\sigma(\mu_{i+1},m)}{\sigma(\mu_i,\mu_{i+1})} < \frac{1}{\lambda}$$

Assume that $\forall\mu \in \Delta$, $\lim_{n\to\infty} \sigma(\mu_n, \mu)$ and $\lim_{n\to\infty} \sigma(\mu, \mu_n)$ exist and are finite. Then, T has a unique fixed point.
In 2020 Lateef [4] generalized the above result as follows:

Theorem 1.2. [4] Let (Δ, Ω, σ) be a complete controlled metric space. Let $T: \Delta \to \Delta$ be such that

$$\Omega(T\mu, T\nu) \leq \gamma\,\Omega(\mu,\nu) + \delta \frac{\Omega(\mu,T\mu)\Omega(\nu,T\nu)}{1+\Omega(\mu,\nu)},$$

$\forall\mu, \nu \in \Delta$, where $\gamma, \delta \in [0, 1)$ such that $\lambda = \gamma + \delta < 1$. For $\mu_0 \in \Delta$, take $\mu_n = T^n\mu_0$. Suppose that,

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\mu_{i+1},\mu_{i+2})\sigma(\mu_{i+1},m)}{\sigma(\mu_i,\mu_{i+1})} < \frac{1}{\lambda}$$

Assume that $\forall\mu \in \Delta$, $\lim_{n\to\infty} \sigma(\mu_n, \mu)$ and $\lim_{n\to\infty} \sigma(\mu, \mu_n)$ exist and are finite. Then, T has a unique fixed point.
In this paper, we generalized the above result in a new type of rational contraction in controlled metric space.

2. Main Results

Theorem 2.1 Let (Δ, Ω, σ) be a complete controlled metric space. Let $T: \Delta \to \Delta$ be such that

$$\Omega(T\mu, T\nu) \leq \gamma \max\left\{\Omega(\mu,\upsilon), \frac{\Omega(\mu,T(\mu))\Omega(\upsilon,T(\upsilon))}{1+\Omega(T(\mu),T(\upsilon))}\right\} + \delta\{\Omega(\mu,T(\mu)) + \Omega(\upsilon,T(\upsilon))\} \tag{2.1}$$

$\forall\mu, \nu \in \Delta$, where $\gamma, \delta \in [0, 1)$ such that $\lambda = \gamma + \delta < 1$. For $\mu_0 \in \Delta$, take $\mu_n = T^n\mu_0$. Suppose that,

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\sigma(\mu_{i+1},\mu_{i+2})\sigma(\mu_{i+1},m)}{\sigma(\mu_i,\mu_{i+1})} < \frac{1}{\lambda}$$

Assume that $\forall\mu \in \Delta$, $\lim_{n\to\infty} \sigma(\mu_n, \mu)$ and $\lim_{n\to\infty} \sigma(\mu, \mu_n)$ exist and are finite. Then, T has a unique fixed point.

Proof.

Let $\mu_0 \in \Delta$. We construct $\{\mu_n\}$ in Δ by $\mu_{n+1} = T\mu_n$, $\forall n \in \mathbf{N}$. If $\exists \mu_n \in \mathbf{N}$ for which $\mu_{n+1} = \mu_n$, then $T\mu_n = \mu_n$. Hence the proof is finished. Now, we assume that $\mu_{n+1} \neq \mu_n$ $\forall n \in \mathbf{N}$. Thus by (2.1), we get

$$\Omega(\mu_n, \mu_{n+1}) = \Omega(T\mu_{n-1}, T\mu_n)$$

$$\leq \gamma \max\left\{\Omega(\mu_{n-1},\mu_n), \frac{\Omega(\mu_{n-1},T(\mu_{n-1}))\Omega(\mu_n,T(\mu_n))}{1+\Omega(T(\mu_{n-1}),T(\mu_n))}\right\}$$

$$+\delta\{\Omega(\mu_{n-1},T(\mu_{n-1})) + \Omega(\mu_n,T(\mu_n))\}$$

$$= \gamma \max\left\{\Omega(\mu_{n-1},\mu_n), \frac{\Omega(\mu_{n-1},\mu_n)\Omega(\mu_n,\mu_{n+1})}{1+\Omega(\mu_n,\mu_{n+1})}\right\}$$

$$+\delta\{\Omega(\mu_{n-1}, \mu_n) + \Omega(\mu_n, \mu_{n+1})\}$$

$$\leq (\gamma+\delta)\Omega(\mu_{n-1}, \mu_n) + \delta\Omega(\mu_n, \mu_{n+1})$$

$$\Omega(\mu_n, \mu_{n+1}) \leq \left(\frac{\gamma+\delta}{1-\delta}\right)\Omega(\mu_{n-1}, \mu_n) = \lambda\Omega(\mu_{n-1}, \mu_n)$$

Similarly,

$$\Omega(\mu_{n-1}, \mu_n) = \Omega\,(T\mu_{n-2}, T\mu_{n-1})$$

$$\leq \gamma \max\left\{\Omega(\mu_{n-2}, \mu_{n-1}), \frac{\Omega\big(\mu_{n-2}, T(\mu_{n-2})\big)\Omega\big(\mu_n, T(\mu_n - 1)\big)}{1 + \Omega\big(T(\mu_{n-2}), T(\mu_{n-1})\big)}\right\}$$

$$+\delta\{\Omega\big(\mu_{n-2}, T(\mu_{n-2})\big) + \Omega\big(\mu_{n-1}, T(\mu_{n-1})\big)\}$$

$$= \gamma \max\left\{\Omega(\mu_{n-2}, \mu_{n-1}), \frac{\Omega(\mu_{n-2}, \mu_{n-1})\Omega(\mu_n, \mu_n)}{1+\Omega(\mu_{n-1}, \mu_n)}\right\}$$

$$+\delta\{\Omega(\mu_{n-2},\ \mu_{n-1}) + \Omega(\mu_{n-1},\ \mu_n)\}$$

$$\leq (\gamma + \delta)\Omega(\mu_{n-2},\ \mu_{n-1}) + \delta\Omega(\mu_{n-1},\ \mu_n)\}$$

$$\Omega(\mu_{n-1}, \mu_n) \leq \left(\frac{\gamma + \delta}{1 - \delta}\right)\Omega(\mu_{n-2},\ \mu_{n-1}) = \lambda\Omega(\mu_{n-2},\ \mu_{n-1})$$

Repeating this process we get,

$$\Omega(\mu_n, \mu_{n+1}) \leq \lambda\Omega(\mu_{n-1},\ \mu_n)$$

$$\leq \lambda^2\Omega(\mu_{n-2},\ \mu_{n-1}) \leq \cdots \leq \lambda^n\Omega(\mu_0,\ \mu_1)$$

Thus, $\Omega(\mu_n, \mu_{n+1}) \leq \lambda^n\Omega(\mu_0,\ \mu_1)$.

For all n, m ∈ **N** (n<m), we have

$$\Omega(\mu_n, \mu_m) \leq \sigma(\mu_n, \mu_{n+1})\Omega(\mu_n, \mu_{n+1}) +\ \sigma(\mu_{n+1}, \mu_m)\Omega(\mu_{n+1}, \mu_m)$$

$$\leq \sigma(\mu_n, \mu_{n+1})\Omega(\mu_n, \mu_{n+1}) +\ \sigma(\mu_{n+1}, \mu_m)\sigma(\mu_{n+1}, \mu_{n+2})\Omega(\mu_{n+1}, \mu_{n+2})$$

$$+\sigma(\mu_{n+1}, \mu_m)\sigma(\mu_{n+2}, \mu_m)\Omega(\mu_{n+2}, \mu_m)$$

$$\leq \sigma(\mu_n, \mu_{n+1})\Omega(\mu_n, \mu_{n+1}) +\ \sigma(\mu_{n+1}, \mu_m)\sigma(\mu_{n+1}, \mu_{n+2})\Omega(\mu_{n+1}, \mu_{n+2})$$

$$+\sigma(\mu_{n+1}, \mu_m)\sigma(\mu_{n+2}, \mu_m)\sigma(\mu_{n+2}, \mu_{n+3})\Omega(\mu_{n+2}, \mu_{n+3})$$

$$+\sigma(\mu_{n+1}, \mu_m)\sigma(\mu_{n+2}, \mu_m)\sigma(\mu_{n+3}, \mu_m)\Omega(\mu_{n+3}, \mu_m)$$

$$\leq$$

$$\vdots$$

$$\leq\ \sigma(\mu_n, \mu_{n+1})\Omega(\mu_n, \mu_{n+1})$$

$$+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mu_j, \mu_m)\right)\sigma(\mu_i, \mu_{i+1})\Omega(\mu_i, \mu_{i+1}) + \left(\prod_{i=n+1}^{m-1}\sigma(\mu_i, \mu_m)\right)\sigma(\mu_{m-1}, \mu_m)\Omega(\mu_{m-1}, \mu_m)$$

$$\leq\ \sigma(\mu_n, \mu_{n+1})\lambda^n\Omega(\mu_0, \mu_1)$$

$$+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(\mu_j, \mu_m)\right)\sigma(\mu_i, \mu_{i+1})\lambda^i\Omega(\mu_0, \mu_1) + \left(\prod_{i=n+1}^{m-1}\sigma(\mu_i, \mu_m)\right)\sigma(\mu_{m-1}, \mu_m)\lambda^{m-1}\Omega(\mu_0, \mu_1)$$

$$= \sigma(\mu_n, \mu_{n+1})\lambda^n\Omega(\mu_0,\mu_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\sigma(\mu_j,\mu_m)\right)\sigma(\mu_i,\mu_{i+1})\lambda^i\Omega(\mu_0,\mu_1)$$

$$\Rightarrow \Omega(\mu_n,\mu_m) \leq \sigma(\mu_n, \mu_{n+1})\lambda^n\Omega(\mu_0,\mu_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\sigma(\mu_j,\mu_m)\right)\sigma(\mu_i,\mu_{i+1})\lambda^i\Omega(\mu_0,\mu_1)$$

Let, $\Gamma_k = \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\sigma(\mu_j,\mu_m)\right)\sigma(\mu_i,\mu_{i+1})\lambda^i\Omega(\mu_0,\mu_1)$

Then we get,

$$\Omega(\mu_n,\mu_m) \leq \Omega(\mu_0,\mu_1)[\lambda^n\sigma(\mu_n,\mu_{n+1}) + (\Gamma_{m-1} - \Gamma_n)]$$

Using $\sigma(\mu,\upsilon) \geq 1$, and ratio test, $\lim_{n\to\infty}\Gamma_n$ exists. Thus $\{\Gamma_n\}$ is Cauchy.

$$\Rightarrow \lim_{n,m\to\infty}\Omega(\mu_n,\mu_m) = 0$$

Thus, $\{\mu_n\}$ is a Cauchy in (Δ, Ω, σ). Let there exists $\mu^* \in \Delta$ such that

$$\lim_{n\to\infty}\Omega(\mu_n,\mu^*) = 0, \qquad (2.2)$$

i.e. $\mu_n \to \mu^*$ as n $\to \infty$. Then from (2.1) and condition 3. of controlled metric space, we get

$$\Omega(\mu^*,T\mu^*) \leq \sigma(\mu^*,\mu_{n+1})\Omega(\mu^*,\mu_{n+1}) + \sigma(\mu_{n+1},T\mu^*)\Omega(\mu_{n+1},T\mu^*)$$

$$= \sigma(\mu^*,\mu_{n+1})\Omega(\mu^*,\mu_{n+1}) + \sigma(\mu_{n+1},T\mu^*)\Omega(T\mu_n,T\mu^*)$$

$$\Omega(\mu^*,T\mu^*) \leq \sigma(\mu^*,\mu_{n+1})\Omega(\mu^*,\mu_{n+1}) +$$

$$\Sigma(\mu_{n+1},T\mu^*)\gamma\max\left\{\Omega(\mu_n,\mu^*),\frac{\Omega(\mu_n,T(\mu_n))\Omega(\mu^*,T(\mu^*))}{1+\Omega(T(\mu_n),T(\mu^*))}\right\} + \delta\{\Omega(\mu_n,T(\mu_n)) + \Omega(\mu^*,T(\mu^*))\}$$

Taking the limit as n $\to \infty$ and by using (2.2), we get a contradiction to $\Omega(\mu^*,T\mu^*) > 0.$ T

Thus $\Omega(\mu^*,T\mu^*) = 0 \Rightarrow \mu^* = T\mu^*$. Hence it is proved.

Corollary 2.1 Let (Δ, Ω, σ) be a complete controlled metric space. Let T: Δ → Δ be such that

$$\Omega(T\mu,T\nu) \leq \gamma\max\left\{\Omega(\mu,\upsilon),\frac{\Omega(\mu,T(\mu))\Omega(\upsilon,T(\upsilon))}{\Omega(\mu,T(\nu))+\Omega(\upsilon,T(\mu))+\Omega(\mu,\upsilon)}\right\} + \delta\{\Omega(\mu,T(\mu)) + \Omega(\upsilon,T(\upsilon))\} \qquad (2.1)$$

∀μ, ν ∈ Δ, where $\Omega(\mu,T(\nu)) + \Omega(\upsilon,T(\mu)) + \Omega(\mu,\upsilon) \neq 0$ and γ, δ ∈ [0, 1) such that $\lambda = \gamma + \delta < 1$ for $\mu_0 \in \Delta$, take $\mu_n = T^n\mu_0$. Suppose that,

$$\sup_{m\geq1}\lim_{n\to\infty}\frac{\sigma(\mu_{i+1},\ \mu_{i+2})\sigma(\mu_{i+1},m)}{\sigma(\mu_i,\ \mu_{i+1})} < \frac{1}{\lambda}$$

Assume that ∀μ ∈ Δ, $\lim_{n\to\infty}\sigma(\mu_n,\mu)$ and $\lim_{n\to\infty}\sigma(\mu,\mu_n)$ exist and are finite. Then, T has a unique fixed point.

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