Int. j. adv. multidisc. res. stud. 2024; 4(5):246-252

Received: 28-07-2024
Accepted: 08-09-2024

International Journal of Advanced Multidisciplinary Research and Studies

ISSN: 2583-049X

Some Applications of Cauchy-Schwarz Inequality on Lightlike Warped Product Submanifolds

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DOI: https://doi.org/10.62225/2583049X.2024.4.5.3224

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Abstract
The well-known Cauchy-Schwarz inequality is a fundamental tool of analysis in Hilbert spaces. It is an upper bound on the inner product between two vectors in an inner product space in terms of the product of the vector norms. It is the most important and widely used inequalities in mathematics. We explore that inequality in case of lightlike warped product hypersurface of lorentzian space form to reach to the existence of minimal screen conformal lightlike hypersurface.

Keywords: Lightlike Warped Product Manifold, Cauchy–Schwarz Inequality, Lightlike Warped Product Isometric Immersions

1. Introduction

Let (M_0, g_0) and (M_i, g_i) be semi-Riemannian manifolds and $\rho_i : M_0 \rightarrow]0, +\infty[$ be positive smooth functions for $i = 1, 2, ..., l$. The multiply warped product $M = M_0 \times_{\rho 1} M_1 \times_{\rho 2} M_2 \times .... \times_{\rho l} M_l$ is the product manifold $M_0 \times M_1 \times M_2 \times .... \times M_l$ furnished with the metric tensor.

$$\langle , \rangle = \pi_0^* \langle , \rangle_0 + \sum_{i=1}^{l} (\rho_i \circ \pi_0)^2 \pi_i^* \langle , \rangle_i \tag{1}$$

where $\pi_i : M \rightarrow M_i$, $i = 0, 1, ..., l$ are the projection morphisms. The functions ρ_i are called the warping functions and (M_0, g_0) the base manifold of the multiply warped product. Each (M_i, g_i), $i = 1, ..., l$ is called a fiber manifold.

- If $l = 1$ then we obtain a singly warped product.
- If $\rho_i(i = 1, ..., l) = c$ where c is a constant then we have a muliple product manifold.
- If for all $i = 0, 1, ..., l$, (M_i, g_i) is a Riemanniann manifold then (M, g) is also a Riemannian multiply warped product manifold. The warped product (M, g) is Lorentzian multiply warped product if (M_i, g_i) $i = 1, ..., l$ are all Riemannian and either (M_0, g_0) is Lorentzian or else (M_0, g_0) is one-dimensional manifold with a negative definite metric $-dt^2$ (t a natural parameter on M_0).
- (M, g) is degenerate (null, lightlike) if (M_0, g_0) is degenerate with $Rad(TM_0)$ of rank r. $Rad(TM)$ still has rank r and all screen structure has dimension $s_0 + \sum_{i=1}^{m} dim(M_i)$. where s_0 is the dimension of any screen structure on M_0.

Towards the end of twentieth century, some authors began to treat the problem concerning isometric immersions of warped product manifolds in semi-Riemannian manifolds. Such as J. D. Moore who in his pioneering work [11] shows under purely intrinsic assumptions that an isometric immersion of a Riemannian product of connected Riemannian manifolds into an Euclidean space must be a product of isometric immersions. Nölker [13] and others authors generalized Moore'results for isometric immersions of Riemannian warped products in Riemannian space forms.

In [3] the autor initiated the study of isometric immersion of lightlike warped product manifols i semi-Riemannian manifold.

In this paper, we explore Cauchy–Schwarz inequality to estabish some inequalities on screen conformal lightlike warped product hypersurface from which we shwow the existence or non- existence of minimal screen conformal hypersuface in case of lorentzian space form.

Let U and V be two vectors in a m-dimensionnal inner product space. The Cauchy–Schwarz inequality inequality states

$$|\langle U,V\rangle|^2 \leq \langle U,U\rangle \langle V,V\rangle$$

In a m-euclidean space $\mathbb{R}$with the standard inner product, the Cauchy-Schwarz inequality becomes

$$\left(\sum_{i=1}^{m} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{m} u_i^2\right)\left(\sum_{i=1}^{m} v_i^2\right)$$

where u_i and v_i are components of U and V respectively.
Take U ($a_1, a_2, \ldots, a_m$) and V (1, 1, , 1) we reach to the following version of The Cauchy–Schwarz inequality.

Lemma 1.1 [12] *If* $a_1, \ldots, a_m$ *are* ($m > 1$)*-real numbers, then*

$$\frac{1}{m}\Big(\sum_{i=1}^{m} a_i\Big)^2 \leq \sum_{i=1}^{m} a_i^2$$

with equality if and only if $a_1 = \ldots = a_m$.

2. Background materials in lightlike submanifold theory

Let (M^{m+1}, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}^{m+2}, \overline{g})$. At each point $p \in M$, the null space of T_pM is a one-dimensional subspace $RadT_pM$ defined by

$$RadT_pM = \{\xi \in T_pM : g(\xi, X) = 0 \,\forall\, X \in \Gamma(T_pM)\}. \tag{2}$$

Since M is a hypersurface we have $TM^{\perp} = RadTM$. A supplementary bundle of $RadTM$ in TM is a rank (m) non-degenerate distribution over M called a screen distribution wchich we denote by $\mathscr{S}(N)$ and we have

$$TM = \mathscr{S}(N) \oplus_{orth} RadTM. \tag{3}$$

The supplementary vector bundle $\mathscr{S}(N)^{\perp}$ of $\mathscr{S}(N)$ in $T\overline{M}$ is called screen transversal bundle and it has rank 2. A lightlike with a specific screen distribution is denoted by $(M, g, \mathscr{S}(N))$. Since $RadTM$ is a lightlike subbundle of $\mathscr{S}(N)^{\perp}$, it is shown in [1] that there exist a unique rank 1 subbundle $tr(TM)$ of $T\overline{M}$ such that for any non zero section ξ of $TM^{\perp}$ on a neighborhood $\mathcal{U} \subset M$, there exist a unique section $N \in tr(TM)$ such that

$$\overline{g}(N, N) = 0 = \overline{g}(X, N),\ \overline{g}(N, \xi) = 1 \,\forall\, X \in \mathscr{S}(N)|_{\mathcal{U}}. \tag{4}$$

Along M we have the following decomposition

$$\begin{aligned} T\overline{M}|_M &= \mathscr{S}(N) \oplus_{orth} (RadTM \oplus tr(TM)) \\ &= TM \oplus tr(TM). \end{aligned} \tag{5}$$

From (4) and (5) one shows that conversely, a choise of a transversal bundle $tr(TM)$ determines uniquely a screen distribution. For this, we choise a null vector field N of the bundle $tr(TM)$ in a neighborhood $\mathcal{U}$ of $\overline{M}$ with $M \subset \mathcal{U}$. The choose of the null transversal field N along M determines both a unique radical vector field ξ and a unique screen distribution $\mathscr{S}(N)$ satisfying (4). In [9], the autors colled the null transversal vector field $N \in \mathcal{U} \supset M$ a *rigging* vector field for M and the unique lightlike vector field $\xi \in M$ such that (4) is satisfied a *rigged* vector field in M. We call the triple (M, g, N) a rigged lightlike hypersurface or a normalized lightlike hypersurface. With respect to (5), for a rigged lightlike hypersurface (M, g, N) a rigged lightlike hypersurface or a normalized lightlike hypersurface.
With respect to (5), for a rigged lightlike hypersurface (M, g, N), for all $X, Y \in \Gamma(TM)$ the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B^N(X,Y)N \tag{6}$$

$$\overline{\nabla}_X N = -A_N X + \tau^N(X)N \tag{7}$$

where B is a local second fundamental form. Let η be a one-form metrically equivalent to N, i.e

$$\eta(X) = \overline{g}(N, X). \tag{8}$$

From (2) we have

$$\nabla_X PY = \nabla^*_X PY + C^N(X, PY)\xi \tag{9}$$

$$\nabla_X \xi = -A^*_\xi X - \tau^N(X)\xi \tag{10}$$

where C is called local screen fundamental form.
It is well known from [4] that a normalized lightlike hypersurface of a semi-Riemannian manifold is called screen conformal if there exists a non-vanishing differential function φ in $\mathcal{U}$ such that

$$A_N = \varphi A^*_\xi \tag{11}$$

and we have

$$C^N(X, PY) = \varphi B^N(X, Y). \tag{12}$$

If φ is a non-zero constant then M is said to be screen homothetic.
From (6, 7, 8, 9) for all $X, Y, Z, W \in \Gamma(TM)$, we have the following Gauss-Codazzi equation

$$\langle R(X,Y)Z, PW\rangle = \langle \overline{R}(X,Y)Z, PW\rangle + B^N(Y,Z)C^N(X,PW) - B^N(X,Z)C^N(Y,PW). \tag{13}$$

Let $\{e_1, ..., e_m\}$ be an orthonormal basis of $\mathscr{S}(N)$. The mean curvature μ of M is defined in [5] as follow

$$\mu = \frac{1}{m}\sum_{i=1}^{m} \epsilon_i B_{ii} \tag{14}$$

with $\epsilon_i = g(e_i, e_i)$, $B_{ii} = B(e_i, e_i)$.

The components of the second fundamental form B and the screen second fundamental form C satisfy

$$\sum_{ij=1}^{m} B_{ij}C_{ji} = \frac{1}{2}\sum_{ij=1}^{m}\left(B_{ij} + C_{ji}\right)^2 - \frac{1}{2}\sum_{ij=1}^{m}\left\{(B_{ij})^2 + (C_{ji})^2\right\} \tag{15}$$

$$\sum_{ij=1}^{m} B_{jj}C_{ii} = \frac{1}{2}\Big\{(\sum_{ij=1}^{m} B_{jj} + C_{ii})^2 - (\sum_{j=1}^{m} B_{jj})^2 - (\sum_{i=1}^{m} C_{ii})^2\Big\}. \tag{16}$$

Let $\pi = span\{X, Y\}$ be a 2-dimensional non-degenerate plane of T_pM. The number

$$K(\pi) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}, \quad \overline{K}(\pi) = \frac{\langle \overline{R}(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2} \tag{17}$$

is called the sectional curvature of π in M and $\overline{M}$ respecively. A 2-plane π of T_pM is called null plane if it is spaned by ξ and X such that $\overline{g}(\xi, X) = 0$ and $\overline{g}(X, X) \neq 0$. The null sectional curvature of π is given in [6] by

$$K_i^{null} = \frac{\langle R(X,\xi)\xi, X\rangle}{\langle X, X\rangle}. \tag{18}$$

Let π_k be a non-null k-plane section of T_pM and $\{e_1, ..., e_k\}$ be any orthonormal basis of π_k. The scalar curvature $\tau(\pi_k)$ of π_k is given by [10]

$$\tau(\pi_k) = \sum_{ij=1}^{k} K(e_i \wedge e_j) \tag{19}$$

where $K(e_i \wedge e_j) = \langle R(e_i, e_j)e_j, e_i\rangle$. If the Riemannian curvature tensor R has the symetry properties we have

$$\tau(\pi_k) = 2\sum_{1\leq i<j\leq k} K(e_i \wedge e_j).$$

3. Applications of Cauchy-Schwarz inequality

Let ($M^m = M_1 \times_\rho M_2$; $g = \pi^*_1(g_1) + (\rho \circ \pi_1)^2 \pi^*_2(g_2)$) be a warped product of two semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2). If we denote ∇ the Levi-Civita connexion of the warped product (M, g), for all $X \in \Gamma(H)$ and $V, W \in \Gamma(V)$ we have [2].

$$\nabla_X V = \frac{X(\rho)}{\rho} V \tag{20}$$

$$R(X,V)W = -\frac{\langle V,W\rangle}{\rho}\nabla_X(grad\rho). \tag{21}$$

Let $\{e_1, ..., em_1, e_{m1+1}, ..., e_m\}$ be an orthonormal frame of (M, g) where $e_1, ..., e_{m1} \in \Gamma(H)$ and $e_{m1+1}, ..., e_m \in \Gamma(V)$. We have

$$\triangle\, \rho = \sum_{i=1}^{m_1}\Big((\nabla_{e_i}e_i)\rho - e_i^2\rho\Big). \tag{22}$$

We recall from [3] the following result.

Theorem 3.1 [3] *Let $f : M = L \times B^{n1} \times_\rho M^{m2}{}_2 \to Q^{m+k}{}_{(c)}$ be an isometric immersion of a global null warped product manifold in a simply-connected, complete Lorentzian manifold $Q^{m+k}{}_{(c)}$ reductive to $H^2 \times_\omega N^{m+k-2}_{(\bar{c})}$, f' and f'' the isometric immersions induced by f on L and $M' = B^{n1} \times_\rho M^{m2}{}_2$ in $Q^{m+k}{}_{(c)}$ respectively. Assume that of the second fundamental form of M satisfy*

$$\pi_{N_*}[\alpha_f(X,Y)] = \alpha_{f''}(\pi_{M'_*}X, \pi_{M'_*}Y)\ \forall X,Y \in \Gamma(TM).$$

Then there exists a global isometry $\Theta : H^2 \times_\omega N^{m+k-2}_{(\bar{c})} \to Q^{m+k}_{(c)}$ and a warped product representation $\psi : V^{n_1+k_1} \times_\sigma N^{m_2+k_2}_{(\bar{c})} \to N^{m+k-2}_{(\bar{c})}$, $k_1 + k_2 = k-1$ such that f is a null warped product of isometric immersions f' and f'' i.e

$$f = \Theta \circ \{f' \times [\psi \circ (f_1'' \times f_2'')]\}.$$

In the following we consoder $M = M_1 \times_\rho M_2$ a lightlike warped product of a (m_1 + 1)- dimensional lightlike manifold with signature (1, 0, m_1) and a m_2-dimensional connected Riemannian manifold isometrically immersed in a Lorentzian manifold $\overline{M}_{(c)}$ of constant sectional curvature c. Taking into account that the ambient is a lorentzian manifold, the induced metric on the screen distribution of M is Riemannian.

Theorem 3.2 Let $M = M_1 \times_\rho M_2$ be a screen conformal rigged lightlike hypersurface of a (m + 2)-dimensional Lorentzian manifold $\overline{M}_{(c)}$ of constant sectional curvature c. Then we have

$$\frac{\triangle\,\rho}{\rho} \le \frac{\varphi m^2\mu^2}{4m_2} + \frac{m_1 c}{2} \tag{23}$$

where $m = m_1 + m_2$ and φ is a positive conformal function. The equality of (23) holds if and only if trace $B|_{\mathcal{H}} =$ trace $B|_{\mathcal{V}}$

Proof: Let $\{e_0, e_1, ..., e_{m1}, e_{m1+1}, ..., e_m\}$ be a quasi-orthonomal frame of $\Gamma(TM)$ such that $\{e_1, ..., e_{m1}, e_{m1+1}, ..., e_m\}$ is a orthonomal frame of $\mathscr{S}(N)$. From (13), (12) and (19) we have

$$\tau_{\mathscr{S}(N)} = \overline{\tau}_{\mathscr{S}(N)} + \varphi m^2\mu^2 - \varphi\sum_{ij=1}^{m} B_{ij}^2. \tag{24}$$

Take

$$\sigma = \tau_{\mathscr{S}(N)} - \frac{1}{2}\varphi m^2\mu^2 - \overline{\tau}_{\mathscr{S}(N)} \tag{25}$$

from (24) we get

$$m^2\mu^2 = 2\Big(\frac{\sigma}{\varphi} + \sum_{i=1}^{m} B_{ii}^2 + 2\sum_{1\le i<j\le m} B_{ij}^2\Big). \tag{26}$$

Let $a_1 = \sum_{i=1}^{m_1} B_{ii}$ Bii and $a_2 = \sum_{j=m_1+1}^{m} B_{jj}$. Using Lemma 1.1 and (26) we get

$$\frac{\sigma}{2\varphi} + \sum_{1\leq i<j\leq m} B_{ij}^2 \leq \sum_{1\leq i<j\leq m_1} B_{ii}B_{jj} + \sum_{m_1+1\leq k<l\leq m} B_{kk}B_{ll}. \tag{27}$$

From the equality

$$\begin{aligned}\sum_{1\leq i<j\leq m}\langle R(e_i,e_j)e_j,e_i\rangle &= \sum_{1\leq i<j\leq m_1}\langle R(e_i,e_j)e_j,e_i\rangle + \sum_{m_1+1\leq k<l\leq m}\langle R(e_k,e_l)e_l,e_k\rangle \\ &+ \sum_{\substack{1\leq i\leq m_1\\ m_1+1\leq k\leq m}}\langle R(e_i,e_k)e_k,e_i\rangle\end{aligned} \tag{28}$$

using (12),(19),(21),(22), and (13) we get

$$\begin{aligned} m_2\frac{\Delta\varphi}{\varphi} &= \frac{1}{2}\tau_{\mathscr{S}(N)} - \frac{1}{2}\overline{\tau}_{\mathscr{S}(N)} + \frac{1}{2}\sum_{\substack{1\leq i\leq m_1\\ m_1+1\leq k\leq m}}\overline{K}(e_i\wedge e_k) + \sum_{1\leq i<j\leq m_1}\varphi B_{ij}^2 \\ &+ \sum_{m_1+1\leq k<l\leq m}\varphi B_{kl}^2 - \sum_{1\leq i<j\leq m_1}\varphi B_{ii}B_{jj} - \sum_{m_1+1\leq k<l\leq m}\varphi B_{kk}B_{ll}.\end{aligned} \tag{29}$$

Since $B_{ij} = 0$ ∀ $i = 1, ..., m_1$
$j = m_1 + 1, ..., m$; (such isometric is mixed totaly geodesic because we have a warped product struture) from (25), (27) and (29) we have

$$\begin{aligned} m_2\frac{\Delta\varphi}{\varphi} &\leq \frac{1}{2}(\tau_{\mathscr{S}(N)} - \overline{\tau}_{\mathscr{S}(N)} + m_1m_2c - \sigma) \\ &= \frac{\varphi m^2\mu^2}{4} + \frac{1}{2}m_1m_2c.\end{aligned}$$

Thus we have $\frac{\Delta\rho}{\rho} \leq \frac{\varphi m^2\mu^2}{4m_2} + \frac{m_1c}{2}$. From (27) we get equality in (23) if and only if trace $B|_{\mathcal{H}}$ = trace $B|_{\mathcal{V}}$.
As consequence of the relation 23 we have

Corollary 3.1 *Let ($M = M_1 \times_\rho M_2$, $g = g_1 + \rho^2 g_2$) be a lightlike warped product of (m_1 + 1)- dimensional lightlike manifold with signature (1, 0, m_1) and a m_2-dimensional connected Riemannian manifold. If ρ is an harmonic function or eigenfunction of the laplacian Δ on M_1 with eigenvalue $\lambda > 0$, then*
(a) there does not exist any screen conformal isometric immersion of M into a (m + 2)- dimensional Lorentzian manifold of negative constant sectional curvature such that

$$\frac{\varphi m^2\mu^2}{2m_2} < |m_1c|$$

(b) there does not exist a minimal screen conformal isometric immersion from M to a (m+2)- dimensional Lorentzian manifold of negative constant sectional curvature.

If the warping function ρ is a non null constant function, the lightlike warped product manifold reduces to a lightlike product one. Since a constant function is harmonic then, from Theorem (3.2), we get the following result.

Theorem 3.3 *Any screen conformal isometric immersion of a lightlike product manifold into euclidian Lorentzian manifold has a positive conformal function.*

Example
An example of the previous result is a lightlike Monge hypersurface of a globally hyperbolic space-time. Let

$$M = \{(x^0, x^1, ..., x^n) \in \mathbb{R}_1^{n+1}, x^0 = F(x^1, x^2, ..., x^n)\}$$

be a Monge hypersurface of $\mathbb{R}_1^{n+1}$ where F is a smooth function of an open of $\mathbb{R}^n$ in $\mathbb{R}$. M is a lightlike hypersurface if the function F satisfies.

$$\sum_{i=1}^{n}(F'_{x_i})^2 = 1.$$

The radical distribution *RadTM* is spanned by the vector field

$$\xi = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} F'_{x^i} \frac{\partial}{\partial x^i}.$$

The transval vector bundle *tr*(*TM*) is spanned by

$$N = -\frac{\partial}{\partial x^0} + \frac{1}{2}\xi.$$

By Proposition 3 in [7] the screen distribution of M is integrable which by Theorem 3 in [8] leads to the decomposition of M in lightlike product $M = C \times M'$ where M' is a complete Riemannian manifold if $\mathbb{R}_1^{n+1}$ is a globally hyperbolic space-time. Finally, from [4] we conclude that the lightlike Monge hypersurface of a globally hyperbolic space-time is sreen conformal lightlike product with positive constant conformal function $\varphi = \frac{1}{2}$.
We use the relation between the product of components of the second fundamental form B and the screen second fundamental form C to give the following theorem.
Theorem 3.4 Let $M = M_1 \times_\rho M_2$ be a screen conformal rigged lightlike hypersurface of a (m + 2)-dimensional Lorentzian manifold $\overline{M}(c)$ of constant sectional curvature c. Then we have

$$\frac{\Delta \rho}{\rho} \leq \frac{m(m-1)}{2m_2}(c + \varphi\mu^2) - \frac{\tau_{\mathscr{S}_1(N)} + \tau_{\mathscr{S}_2(N)}}{2m_2} \tag{30}$$

where

$$\mathscr{S}_1(N) = \{X \in \mathscr{S}(N) : X \in \Gamma(\mathcal{H})\}$$

$$\mathscr{S}_2(N) = \{V \in \mathscr{S}(N) : V \in \Gamma(\mathcal{V})\}.$$

Proof.
Consider (13), (16) and (12) we have

$$\begin{aligned} \frac{2m_2 \,\Delta \rho}{\rho} + \tau_{\mathscr{S}_1(N)} + \tau_{\mathscr{S}_2(N)} &= \overline{\tau}_{\mathscr{S}(N)} + \frac{1}{2}(\sum_{ij=1}^{m} B_{jj} + C_{ii})^2 - \frac{1}{2}(\sum_{j=1}^{m} B_{jj})^2 \\ &\quad -\frac{1}{2}(\sum_{i=1}^{m} C_{ii})^2 - \sum_{ij=1}^{m} B_{ij}C_{ji} \\ &= m(m-1)c + \frac{(2\varphi+1)}{2}m^2\mu^2 - \frac{1}{2}m^2\mu^2 - \sum_{ij=1}^{m}\varphi B_{ij}^2 \\ &\leq m(m-1)c + \varphi m^2\mu^2 - \varphi\sum_{i=1}^{m} B_{ii}^2. \end{aligned} \tag{31}$$

By Lemma 1.1 we have $-\varphi m\mu^2 \geq -\varphi\sum_{i=1}^{m} B_{ii}^2$ and (30) follows.

4. Conflicts of Interest
The authors declare no conflicts of interest regarding the publication of this paper.

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