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Motion of a Particle in the Field of a Massive Rotating Body

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Abstract
The geometry of the two-dimensional space surrounding the axis of rotation passing through the center of mass of a spheroidal body is described. Only a cylindrically symmetric one-sheet hyperboloid can be such a space. It is shown that sections of this space by planes passing through the center are divided into two groups by the angle between the cutting plane and the axis of rotation. The boundary value of the angle is determined by the ratio of the principal radii of the spheroid. The trajectory formed by the section by a plane whose angle with the axis of rotation is less than the boundary value is a hyperbola. Otherwise, the trajectory is a closed regular ellipse. If the angle is equal to $\frac{\pi}{2}$, the ellipse becomes a circle whose radius is the maximum radius of the spheroid. If a particle moves along a hyperbola, its trajectory is not closed. An ellipsoidal trajectory is a trajectory of a planet.

Keywords: Kinetic Moment of a Massive Body, Geometry of Two-Dimensional Encircling Space, One-sheet Hyperboloid, Asymptotic Cone, Section Plane, Trajectory, Hyperbola, Ellipse

1. Formulation of the problem

In the work [1] it is shown that three-dimensional Riemannian space is always conformally Euclidean, i.e. the square of the length element of a curve can always be represented in the form:

$$ds^2 = [1 + G(\mathbf{r})][(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \quad (1)$$

Here $(x^1, x^2, x^3) -$ are the coordinates of the local orthogonal frame, $\mathbf{r}$ - the coordinate of the point in three-dimensional space, $[\,1 + G(\boldsymbol{r})]$ - are the diagonal components of the diagonalized metric tensor. In Euclidean space $G(\mathbf{r}) \equiv 0$, therefore, this function can be considered as a gravitational potential. If we consider the neighborhood of a body whose mass is concentrated in the center of mass, then we can determine:

$$G(\mathbf{r}) = \frac{M}{|\mathbf{r}|} = \frac{M}{\sqrt{x^2 + y^2 + z^2}} \quad (2)$$

This formula describes a spherically symmetric field.

The gravitational field can also be created by several massive bodies, arbitrarily located in space. In any case, it can be described by equipotential surfaces. Then the inertial motion can be defined by the law: "A material point maintains a state of rest or motion along a geodesic line of an equipotential surface with a constant speed until an external force acts on it."

In this paper we will consider the case where a body defining the geometry of the space surrounding it is characterized not only by a mass concentrated in the center of mass, but also by a vector of kinetic momentum **L** that pass through the center of mass and perpendicular to the plane of rotation, i.e. fixing the axis of rotation [2]. But if in three-dimensional space there is a constant rectilinear axis, then Riemannian geometry can be defined only in curved two-dimensional space - a surface that encloses the axis of rotation. Often, when describing the geometry of this surface and the kinematics of the motion of a particle on it, the surface is considered as embedded in three-dimensional space.

In the second section of this work we will consider the geometry of two-dimensional space. Everything presented in this section is known, mathematically proven facts [3].

The third section examines the inertial motion of a particle in the two-dimensional space described above. Possible trajectories and velocities along them, determined by the initial conditions, are considered.

2. Geometry of two-dimensional space, determined by fixing the axis of rotation of the spheroid

For simplicity, we will restrict ourselves to considering the motion of a particle in the vicinity of a star that rotates around an axis passing through its center of mass and is an oblate spheroid. Its principal radii are: $R_\perp$ on the plane perpendicular to the rotation axis and $R_\parallel$ on the rotation axis. Otherwise, we will assume that the star and the particle are homogeneous. The rotation axis passes through the center of the spheroid perpendicular to the plane of greatest cross-section. Further, we will consider this axis to be the coordinate axis of the variable z. In this section of the work we consider only the geometry of space. Kinematics, that is, the inertial motion of a small particle in it, is considered in the third section of this article.

The axis of rotation violates the isotropy of three-dimensional space. This direction is excluded from arbitrary transformations of the coordinate system. Thus, the space of particle motion becomes a two-dimensional space. Let us consider the geometry of this space.

The particle's motion surface must be cylindrically symmetrical relative to the axis, but not cylindrical. The distance of the surface points from the axis must increase with distance from the spheroid's center. It must also be mirror-symmetrical relative to the plane of the spheroid's smallest section. There must be no umbilical points (poles) on the surface, at which it is impossible to determine coordinates.

There is only one surface that, under certain conditions for the parameter values, satisfies all these requirements - a single-sheet hyperboloid. The general equation of a one-sheet hyperboloid [3]:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad (3)$$

To satisfy the requirement of cylindrical symmetry $a^2=b^2=R_\perp^2$ and $c^2=R_\parallel^2$. Then equation (3) has the form:

$$\frac{\rho^2}{R_\perp^2}-\frac{z^2}{R_\parallel^2}=1, \qquad (4)$$

where the coordinate $\rho(z)=\sqrt{x^2+y^2}$ is the radius of the circle of the section of the surface by a plane perpendicular OZ in the point z.

When cutting the space with a plane in which lies OZ, we obtain two hyperbolas symmetrically located with respect to OZ. The shortest distance between two symmetric points on the lines of the hyperbolas is $2R_\perp$. These points are called the vertices of the hyperbolas. Obviously $R_\perp$ this is the smallest possible value ρ, that is, with a value $\rho=R_\perp$ the hyperboloid is only touches the surface of the rotating spheroid. The plane perpendicular OZ at the point $z=0$ intersects the hyperboloid along a circle connecting the vertices of all the hyperbolas that form the hyperboloid. This plane is called the diametral plane.

A pair of hyperbolas lying on the same plane passing through the axis OZ has two asymptotes - straight lines that intersect at the origin and are described by the equations $z=\left(\frac{R_\parallel}{R_\perp}\right)\rho, z=-\left(\frac{R_\parallel}{R_\perp}\right)\rho$. Here $R_\parallel$ − is the coordinate of the intersection with the asymptote of the tangent to the hyperbola at its vertex. Determining the value of this parameter completely determines the generating hyperbola.

When rotating around the axis OZ the asymptotes form a conical surface (asymptotic cone), which is located in the cavity of the hyperboloid. All generatrix of the conical surface intersect at the point $z=0$. At $\rho \gg R_\perp$ the conical surface is close to the surface of the hyperboloid.

If we shift the plane of the section by dz, then the radius of the circle will change by $d\rho$. In this case $d\rho \perp dz$, therefore

$$dg^2=d\rho^2+dz^2. \qquad (5)$$

Here dg is the differential of a line which, when rotating around an axis, describes the surface of a single-sheet hyperboloid.

3. Kinematics and dynamics of a particle in a two-dimensional enveloping space

In the kinematics of classical mechanics, the trajectory of a material point in the absence of forces acting on it is determined by the initial point and the initial velocity vector, which is preserved during the motion. The two-dimensional space of a one-sheet hyperboloid, which we will call envelope, is not affine. Therefore, it is impossible to introduce the concepts of rectilinear motion and vector in it.

A certain direction in three-dimensional space is fixed. But this direction is in no way connected with the particle whose motion is being considered. Fixing this direction is an absolute.

To describe the motion of a particle, it is necessary to define a system of lines covering the entire surface. These lines may intersect or touch, but must not coincide on a finite segment. Then each such line can be an inertial trajectory of the particle. The envelope space is a special case of two-dimensional spaces well studied by geometers. The system of inertial trajectories for the envelope space proposed by the author below is based on the theory of surfaces [3]. For other types of two-dimensional spaces, inertial trajectories must be defined differently.

Let us consider all circular conical surfaces with an axis OZ and a vertex at a point $z=0$.. Each of them is characterized by the angle α between the cone surface and the axis OZ. Each such surface can be uniquely represented as an asymptotic cone of a cylindrically symmetric hyperboloid with parameters $a=b=R_\perp$, c [(see formula (3)]. The angle α for each cone is determined by the equation $tg(\alpha)=\frac{R_\perp}{c}$. For the asymptotic cone of the surface under consideration, we denote this angle φ:

$$tg(\varphi)=\frac{R_\perp}{R_\parallel} \qquad (6)$$

Let us consider the sections of these cones by planes that pass through a certain diameter of the diametral plane and differ in the angle β between the perpendicular to the plane in the center $z=0$ and the axis OZ.

If $\beta < \varphi$, then for the corresponding plane in formula (6) instead $R_{\parallel}$ of will be $c > R_{\parallel}$. Therefore, the section of the surface by this plane is also two hyperbolas, but with a smaller angle at the apex of the asymptotic cone. These trajectories describe the arrival of a particle from afar, its contact with the spheroid at a distance $R_{\perp}$ from the center and symmetrical relative to the diametrical plane removal to infinity.

If $\beta > \varphi$, then for the corresponding plane in formula (6) instead $R_{\parallel}$ of will be $c < R_{\parallel}$ In this case, the section of the hyperboloid surface is an ellipse, the small diameter of which lies on the diametral plane, and the large diameter depends on the angle. Due to rotational symmetry, the orientation of the diameters on the diametral plane is unimportant. The condition of the limited trajectory explains the reason that the orbits of the planets lie near the plane of the ecliptic. The same reason, perhaps, explains the fact that galaxies have the shape of a disk.

To describe the motion along an ellipse, we need to introduce time as a parameter to describe the curve. The major and minor diameters of the ellipse are orthogonal, so we can write.

$$dl^2 = dx^2 + dy^2 \quad (7)$$

Here $dx, \mathrm{d}y -$ are the differentials of orthogonal coordinates, $\mathrm{d}l$ - is the differential of the arc of the ellipse. We divide the elements of the arc and coordinates by and introduce the definition of velocity. We obtain.

$$v_l^2 = v_x^2 + v_y^2 \quad (8)$$

This is a generalization of the law of inertia to curvilinear coordinates: "When moving along a curve, the modulus of velocity (kinetic energy) is conserved."

4. References

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