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Short Note: Exploring Ideals in Graph Theory

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Abstract
We are re-evaluating the novel concept of 'ideal' within the context of graph theory. The concept of an 'ideal' is well-established in the fields of topology and algebra, having been extensively researched by numerous scholars due to its mathematical significance (see references) [2-6]. Tree-width is a well-known graph width parameter. In our discussion, we explore the relationship between tree-width and ideal. Overall, this exploration contributes to a deeper understanding of these concepts and their relevance in various mathematical and logical contexts.

Keywords: Tree-width, Ideal, Ultrafilter

1. Notation in this paper

We introduce some notations in this concise paper, drawing inspiration from reference [1].

In this concise paper, a set is defined as a collection of distinct elements or objects, treated as a single entity and typically denoted by curly braces. A subset is a set whose elements are all contained within another set. Boolean algebra $(X, \cup, \cap)$ represents a mathematical structure comprised of a set X, along with operations of union ($\cup$) and intersection ($\cap$), which satisfy specific axioms. Furthermore, this paper focuses on finite sets.

In this paper, we use expressions like $A \subseteq X$ to indicate that A is a subset of X, $A \cup B$ to represent the union of two subsets A and B (both of which are subsets of X), and $A = \emptyset$ to signify an empty set. Specifically, $A \cap B$ denotes the intersection of subsets A and B. Similarly, $A \setminus B$ represents the difference between subsets A and B. The powerset of a set A, denoted as 2^A, is the set of all possible subsets of A, including the empty set and A itself.

In this concise paper, we adopt certain notations inspired by reference [1]. In this paper, we deal with undirected and finite graphs. For a graph G, $V(G)$ denotes the set of vertices, while $E(G)$ represents its set of edges. When we express G as (V, E), it indicates that the graph G is defined by two sets: V for vertices and E for edges.

Furthermore, a separation in a graph G refers to a pair of subgraphs (A, B) that adhere to the following conditions:

- The union of vertex sets, $V(A) \cup V(B)$, equals $V(G)$, where $V(X)$ designates the vertex set of X.
- The intersection $V(A) \cap V(B)$ is non-empty yet minimal, implying no subsets of A and B can qualify as a separate separation. The order of a separation (A, B) is determined by $|V(A) \cap V(B)|$, which quantifies the shared vertices between A and B.

2. Ideal on Boolean Algebras

We provide an explanation of Ideals in Boolean Algebras.

Definition 1: In a Boolean algebra $(X, \cup, \cap)$, a set family $I \subseteq 2^X$ satisfying the following conditions is called an ideal on the carrier set X.

(IB1) $A, B \in I \Rightarrow A \cup B \in I$ (Closure under Union),

(IB2) $B \in I, A \subseteq B \subseteq X \Rightarrow A \in I$ (Closure under Superset),

(IB3) X is not belong to I (Exclusion of the Universal Set).

In a Boolean algebras $(X, \cup, \cap)$, A maximal ideal satisfies the following axiom (IB4):

(IB4) $\forall A \subseteq X$, *either* $A \in I$ *or* $X / A \in I$

3. Ideal on the graph

Next, we provide an explanation of ideal on the graph. The definition of a *G*-Ideal on the graph is given below. We naturally extend the definition from Boolean algebras to graphs. Note that in this paper, we utilize the natural number *k*.

Definition 2: Let *G* be a graph and *k* be a natural number. A *G*-Ideal of order *k* is a family *I* of separations of *G* satisfying the following conditions.

(I0) The order of all separations $(A, B) \in I$ is less than *k*.

(I1) $(A_2, B_2) \in I$, $A_1 \subseteq A_2$, (A_1, B_1) of *G* of order less than *k* $\Rightarrow (A_1, B_1) \in$ I,

(I2) $(A_1, B_1) \in$ I, $(A_2, B_2) \in$ I, $(A_1 \cup A_2, B_1 \cap B_2)$ of *G* of order less than *k*

$\Rightarrow (A_1 \cup A_2, B_1 \cap B_2) \in$I,

(I3) *If* $V(A) = V(G)$, *then* $(A, B) \notin I$.

This is a theorem that illustrates the characteristics of a maximal *G*-ideal.

Theorem 1: Let *G* be a graph and *k* be a natural number. If a G-*ideal I on graph* is a maximal then *I* satisfies following axiom.

(I4) For all separations *(A, B)* of *G* of order less than *k*, either $(A, B) \in I$ or $(B, A) \in I$.

Proof. Suppose there exists an (A, B) such that the order of (A, B) is less than *k*, and neither *(A, B)* nor *(B, A)* belongs to *I*. Among all such *(A, B)*, choose the one with the smallest order, and denote it as I_A. I_A includes $I \cup (A, B)$ and forms a minimal family satisfying axioms (I0), (I1), (I2), and (I3). Similarly, let I_B be the set that includes $I \cup (B, A)$ and forms a minimal family satisfying axioms (I0), (I1), (I2), and (I3).

From the maximality of *I*, both I_A and I_B must contain an separation *(C, D)* such that $V(C) = V(G)$. Therefore, there exist E_A in I and E_B in I.

Given that I_A includes $I \cup (A, B)$ and I_B includes $I \cup (B, A)$, we know that they contain more separations than I and they still respect the axioms (I0) to (I3). Remember, we're trying to show a contradiction from the assumption that neither *(A, B)* nor *(B, A)* belongs to *I*.

From the definition of I_A and I_B, both I_A and I_B must have a separation *(C, D)* such that $V(C) = V(G)$ due to the given condition. But, based on axiom (I3), no separation with $V(C) = V(G)$ should belong to the ideal. Thus, there must be some separation in *I* that contradicts with the inclusion of *(A, B)* or *(B, A)* belongs to the ideal.

Given that E_A is in *I* an E_B is in *I*, it implies that combining them with either *(A, B)* or *(B, A)* would result in a separation of the form *(C, D)* where $V(C) = V(G)$.

Now, using the properties of separations, we deduce the following:

1. Consider the separations E_A and *(A, B)*. By axiom (I2), their combination $(E_A \cup A, E_B \cap B)$ should be a valid separation of the graph *G*. But by our above reasoning, this implies $V(E_A \cup A) = V(G)$, which is a contradiction to axiom (I3).
2. Similarly, for the separations E_B and *(B, A)*, their combination *(*$E_B \cup B$*, E_A* $\cap$ *A)* would also yield $V(E_B \cup B) = V(G)$, contradicting axiom (I3) of *G*-ideal.

From the two contradictions deduced from the assumption that neither *(A, B)* nor *(B, A)* belongs to *I*, we can conclude that for every separation *(A, B)* of *G* of order less than *k*, either *(A, B)* belongs to *I* or *(B, A)* belongs to *I*. Thus, the theorem is proven.

4. Relation to Tree-decomposition

Here, we briefly explain the relationship between 'ideal' and the graph width parameter.

Graph Width parameters pertain to metrics derived from tree-like structures, commonly referred to as graph decompositions. Among these metrics, one of notable significance is the tree width, which has been demonstrated to serve as a pivotal metric in delineating the intricacy of diverse mathematical entities, encompassing graphs and matroids (refer to) [7-20].

G-Ultrafilter is a concept in graph theory that is the complement of a maximal G-ideal. G-Ultrafilter possesses the following property:

Theorem 2 [1]**:** Let *G* be a graph and *k* be a natural number. If there exists a *G*-Ultrafilter of order $k - 1$, then the tree-width is at least *k*.

Therefore, a maximal *G*-ideal also possesses the following property:

Theorem 3: Let *G* be a graph and *k* be a natural number. If there exists a maximal G-ideal of order $k - 1$, then the tree-width is at least *k*.

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