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Short Survey of Obstruction to Rank-width and Linear Rank Width

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Abstract
Rank-width, as outlined in [15], stands as a prominent graph width parameter in the realm of graph theory. Within this document, we delve into obstructions related to Rank-width. While it may not present groundbreaking novelty, this concise paper serves the purpose of organizing information.

Keywords: Rank Width, Ultrafilter, Tangle, Linear Rank Width

1. Short Introduction

Graph theory is a fundamental branch of mathematics that focuses on the study of networks formed by nodes and edges, examining their paths, structures, and properties. The "graph width parameter" measures the width of a graph, typically representing the maximum width across all cuts or layers in a hierarchical decomposition. Recently, there has been a notable surge in research interest concerning width parameters in both graph theory and algebraic contexts, as evidenced by references [1-13, 22-40, 51]. These width parameters are closely linked to metrics rooted in tree-like structures, commonly referred to as graph decompositions.

In this short survey, we delve into obstructions related to rank-width, a measure of graph complexity using a tree-like decomposition and rank functions of vertex subsets. While it may not present groundbreaking novelty, this concise paper serves the purpose of organizing and consolidating existing information.

2. Preperation: Notations and Definitions

In this section, we explain about notations and definitions in this paper. $V(G)$ represents the set of vertices (nodes) in a graph G, $E(G)$ represents the set of edges in the same graph G, and $G=(V,E)$ signifies that G is a graph defined by a pair of sets, V for vertices and E for edges. Additionally, in this short paper, we utilize the natural number k, i, *and* j.

In this short paper, we use expressions like $A \subseteq X$ to indicate that A is a subset of X, $A \cup B$ to represent the union of two subsets A and B, both of which are subsets of X, or $A = \emptyset$ to signify an empty set. Specifically, $A \cap B$ denotes the intersection of subsets A and B. A similar logic applies to $A \setminus B$. The powerset of a set A, denoted as 2^A, is the set of all possible subsets of A, including the empty set and A itself.

And we provide an explanation of Filters in Boolean Algebras. The definition of a filter in a Boolean algebra $(X, \cup, \cap)$ is given below. We extend the definition of Filters in a Boolean algebra $(X, \cup, \cap)$ to establish a dual relationship with rank-width.

Definition 1: In a Boolean algebra $(X, \cup, \cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.

(FB1) $A, B \in F \Rightarrow A \cap B \in F$,

(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,

(FB3) $\emptyset$ is not belong to F.

In a Boolean algebras $(X, \cup, \cap)$, a maximal filter is called an ultrafilter and satisfies the following axiom (FB4): (FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

Ultrafilter is a fundamental mathematical concept that has been extensively studied in various fields [52-58]. It is fascinating that seemingly unrelated concepts become connected to rank-width and linear-rank-width when the conditions of rank functions, discussed later, are applied.

3. Obstruction to Rank-width

We provide an explanation concerning the concepts of Rank-width and its obstructions.

Rank-width, as outlined in [15], stands as a prominent graph width parameter in the realm of graph theory. It is characterized by the smallest integer, denoted as k, which allows a graph to be subdivided into a tree-like structure through a recursive partitioning of its vertex set, ensuring that each division results in a matrix with a rank no greater than k. Notably, Rank-width shares a close connection with clique-width [17]. The substantial volume of research devoted to Rank-width, as evident from publications encompassing the range [14-21, 41-50], underscores its significance in the field.

Given that rank-width is a special case of branch-width associated with a symmetric submodular function, we can leverage the theory developed for symmetric submodular functions. Specifically, we can employ concepts known as a "tangle" and an "Ultrafilter" to ascertain the presence of a substantial rank-width in a graph.

To begin, let's delve into the concept of the cut-rank function. The definition of the cut-rank function is provided below.

Definition 2 [15, 17]: For a graph G and a subset A of the vertex set $V(G)$, we can define $\rho(A)$ as the rank of a $|A| \times |V(G) \setminus A|$ 0-1 matrix, denoted as X_A, defined over the binary field. The entry of X_A in the i-th row and j-th column is set to 1 if and only if the i-th vertex in A is adjacent to the j-th vertex in $V(G) \setminus A$.

The cut-rank function ρ exhibits the following inequalities. Note that cut-rank function possesses symmetry and submodularity.

(1) $\rho(A) = \rho(V(G)\setminus A)$ for all $A \subseteq V(G)$,

(2) $\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$ for all $A, B \subseteq V(G)$.

By utilizing the aforementioned cut-rank function, it becomes possible to define a Tangle as follows. Using a similar approach, an Ultrafilter with a deep connection to the Tangle can also be defined.

Definition 3 [17]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a ρ-tangle of order $k+1$ is a set T of subsets of $V(G)$ satisfying the following three axioms.

(T1) For $A \subseteq V(G)$, if $\rho(A) \leq k$, then either $A \in T$ or $V(G) \setminus A \in T$.

(T2) If $A, B, C \in T$, then $A \cup B \cup C \neq V(G)$.

(T3) For all $v \in V(G)$, $V(G) \setminus \{v\} \notin T$.

Definition 4 [13]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a set S of subsets of $V(G)$ is called an order $k + 1$ ρ-ultrafilter if it satisfies the following conditions.

(F1) For any $A, B \in S$ and $\rho(A \cap B) \leq k$, then $A \cap B \in S$.

(F2) For any $A \in S$ and $A \subset B \subseteq V(G)$, if $\rho(B) \leq k$, then $B \in S$.

(F3) $\emptyset$ is not belong to S.

(F4) For any $A \subseteq V(G)$, if $\rho(A) \leq k$, either A ∈ S or $(V(G) / A) \in S$.

Based on previous research results in branch-width, it is possible to derive the following duality theorem.

Theorem 5 [11, 13]: Let ρ be a cut-rank function, and let k be a natural number. For an integer k, a graph G has a ρ-tangle (ρ-ultrafilter) of order k if and only if its rank-width is at least k.

4. Obstruction to Linear-Rank-width

We provide an explanation of the concepts related to Linear-Rank-width and its associated obstructions. Numerous articles within the field of parameter studies address linear width parameters, with a particular focus on width parameters constrained to caterpillars. Similarly, just like Rank-width, there exists a linear version of the graph width parameter known as Linear rank-width.

Linear rank-width is essentially a linear variant, often referred to as the caterpillar version, of rank-width [15, 17]. Just as research on rank-width holds significance, the study of Linear-Rank-width is equally important.It allows for the definition of an obstruction called "ρ-obstacle", "ρ-linear-tangle", and "ρ-single-ultrafilter" which is outlined below.

Definition 6 [6, 17]: Let ρ be a cut-rank function, and let k be a natural number. For an integer k, ρ-obstacle of order $k + 1$ is a set O of subsets of $V(G)$ satisfying the following three axioms.

(O1) For all $A \in O$, $\rho(A) \leq k$.

(O2) If $A \subseteq B$, $B \in O$ and $\rho(A) \leq k$, then $A \in O$.

(O3) If $A \cup B \cup C = V(G)$, $A \cap B = \emptyset$, $\rho(A) \leq k$, $\rho(B) \leq k$ and $|C| \leq 1$, then exactly one of A and B is in O.

Definition 7 [7]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a set S of subsets of $V(G)$ is called an order $k + 1$ ρ-single-ultrafilter if it satisfies the following conditions.

(S1) For any $A \in S$, $v \in V(G)$, if $\rho(\{v\}) \leq k$ and $\rho(A \cap (V(G)/\{v\})) \leq k$, then $A \cap (V(G)/\{v\}) \in S$.

(S2) For any $A \in S$ and $A \subset B \subseteq V(G)$, if $\rho(B) \leq k$, then $B \in S$.

(S3) $\emptyset$ is not belong to S.

(S4) For any $A \subseteq V(G)$, if $\rho(A) \leq k$, either $A \in S$ or $(X/A) \in S$.

Definition 8 [10]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a ρ-linear-tangle of order $k+1$ is a set L of subsets of $V(G)$ satisfying the following three axioms.

(L1) For $A \subseteq V(G)$, if $\rho(A) \leq k$, then either $A \in L$ or $V(G) \setminus A \in L$.

(L2) If $A, B \in L$, $\rho(\{v\}) \leq k$, then $A \cup B \cup \{v\} \neq V(G)$.

(L3) For all $v \in V(G)$, $V(G) \setminus \{v\} \notin L$.

Based on previous research results in linear-branch-width, it is possible to derive the following duality theorem.

Theorem 9 [7, 10, 17]**:** Let ρ be a cut-rank function, and let k be a natural number. A graph G has a ρ-obstacle (ρ-linear-tangle, ρ-single-ultrafilter) of order k if and only if its linear rank-width is at least k.

5. Future tasks: Obstruction to Hypertree-width

We are also very interested in the notion of hypertree width [23] of hypergraphs. Like other graph width parameters, research is actively progressing in the field of hypertree width. Similar to rank-width, there has been a wealth of research results in the field of hypertree width, as evidenced by publications [22-31].

An concept known to have a deep connection with hypertree width is the Hypertangle number. Based on previous research results, it wouldn't be surprising if there were a concept similar to the HyperUltrafilter number, just as the Hypertangle number exists. We plan to continue our investigation in this direction.

6. Future tasks: A proximity and a cluster on a proximity spaces

In the future, we will discuss about relationship between a proximity / a cluster / a semi-ultrafilter on a proximity spaces and rank-width / linear rank-width.

Proximity spaces are a generalization of topological spaces where the concept of "closeness" or "nearness" between sets is defined. They consist of a set along with a proximity relation that satisfies specific axioms, allowing the study of continuity and convergence without relying on a metric (cf. [59-68]).

The definition of a proximity and a cluster and on proximity spaces is provided below.

Definition 10 [59]**:** Let X be a nonempty set. A relation δ on the family $P \in 2^X$ of all subsets of a set X is called a proximity on X if δ satisfies the following conditions:

(P1) If $A\delta B$, then $B\delta A$;

(P2) $A\delta(B \cup C)$ if and only if $A\delta B$, or $A\delta C$

(P3) $X \bar{\delta} \emptyset$;

(P4) $\{x\}\delta\{x\}$ for each $x \in X$

(P5) If $A\bar{\delta}B$, then there exists $E \in P$ such that $A \bar{\delta} E$ and $X \setminus E \bar{\delta} B$

The pair (X, δ) is called a proximity space. We should use the notation $(A, B) \in \delta$ or $(A, B) \notin \delta$ when the sets A and B are either near each other or not, but we shall simply write $A\delta B$ or $A\bar{\delta}B$,

Definition 11 [60]**:** A nonempty family of nonempty subsets of X denoted by C is called a cluster in proximity space (X, δ) if σ satisfies the following conditions:

(C1)For all $A, B \in C$, $A\delta B$;

(C2)$(A \cup B) \in C$ if and only if $A \in C$ or $B \in C$;

(C3)$A\delta B$ for each $B \in C$ implies $A \in C$.

The definition of a semi-ultrafilter and on proximity spaces is provided below.

Definition 12 [65]**:** A nonempty family (cluster) of nonempty subsets of X denoted by F is called a semi-ultrafilter in proximity space (X, δ) if σ satisfies the following conditions:

(C3) $A\delta B$ for each $B \in F$ implies $A \in F$

(C4) Let p and q both belong to X. If $\{p\}$ and $\{q\}$ belong simultaneously to the cluster F, then $p = q$.

(C5) $A \in F$, $A \subseteq B \subseteq X \Rightarrow B \in F$,

(C6) A is a subset of X and A meets each B from F, *then A also belongs to F.*

We will examine the relationships with rank-width and branch-width by incorporating the conditions of rank functions and symmetric submodular functions into the above concepts.

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8. Conflict of Interest Statement

The author declares no conflicts of interest.

9. References

1. Liu Chun-Hung. A global decomposition theorem for excluding immersions in graphs with no edge-cut of order three. Journal of Combinatorial Theory, Series B. 2022; 154:292-335.
2. Bodlaender Hans L. Treewidth: Algorithmic techniques and results. International Symposium on Mathematical Foundations of Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg, 1997.
3. Kanté Mamadou Moustapha, *et al*. Obstructions for matroids of path-width at most k and graphs of linear rank-width at most k. Journal of Combinatorial Theory, Series B. 2023; 160:15-35.
4. Diestel Reinhard, *et al*. Duality and tangles of set separations. arXiv preprint arXiv:2109.08398, 2021.
5. Fujita Takaaki. Proving Maximal Linear Loose Tangle as a Linear Tangle. Asian Research Journal of Mathematics. 2024; 20(2):48-54.
6. Fedor V Fomin, Dimitrios M Thilikos. On the monotonicity of games generated by symmetric submodular functions. Discrete Applied Mathematics. 2003; 131(2):323335.
7. Fujita Takaaki. Matroid, Ideal, Ultrafilter, Tangle, and so on: Reconsideration of Obstruction to linear decomposition, 2023. arXiv preprint arXiv:2309.09199.
8. Fujita Takaaki. Alternative Proof of Linear Tangle and Linear Obstacle: An Equivalence Result. Asian

Research Journal of Mathematics. 2023; 19(8):61-66.
9. Fujita Takaaki. Ultrafilter in Graph Theory: Relationship to Tree-decomposition, 2023.
10. Daniel Bienstock. Graph searching, path-width, tree-width and related problems (a survey). In Reliability of Computer and Communication Networks, Vol. 5 of DIMACS. Series in Discrete Mathematics and Theoretical Computer Science, 1989, 33-50.
11. Neil Robertson, Paul D Seymour. Graph minors. X. Obstructions to tree-decomposition. Journal of Combinatorial Theory, Series B. 1991; 52(2):153-190.
12. Sang-il Oum, Paul Seymour. Testing branch-width. Journal of Combinatorial Theory, Series B. 2007; 97(3):385-393.
13. Fujita Takaaki. Reconsideration of tangle and ultrafilter using separation and partition, 2023. arXiv preprint arXiv:2305.04306.
14. Oum, Sang-il. Approximating rank-width and clique-width quickly. ACM Transactions on Algorithms (TALG). 2008; 5(1):1-20.
15. Oum Sang-il. Rank-width and vertex-minors. Journal of Combinatorial Theory, Series B 95.1, 2005, 79-100.
16. Kanté, Mamadou Moustapha, Michael Rao. The rank-width of edge-coloured graphs. Theory of Computing Systems. 2013; 52(4):599-644.
17. Oum Sang-il. Rank-width: Algorithmic and structural results. Discrete Applied Mathematics. 2017; 231:15-24.
18. Lee Choongbum, Joonkyung Lee, Sang-il Oum. Rank-width of random graphs. Journal of Graph Theory. 2012; 70(3):339-347.
19. Oum, Sang-il. Rank-width and well-quasi-ordering. SIAM Journal on Discrete Mathematics. 2008; 22(2):666-682.
20. Kwon O-joung, Michał Pilipczuk, Sebastian Siebertz. On low rank-width colorings. European Journal of Combinatorics. 2020; 83:103002.
21. Ganian Robert. Thread graphs, linear rank-width and their algorithmic applications. Combinatorial Algorithms: 21st International Workshop, IWOCA 2010, London, UK, July 26-28, 2010, Revised Selected Papers 21. Springer Berlin Heidelberg, 2011.
22. Safari MohammadAli. Hyper-T-width and hyper-D-width: Stable connectivity measures for hypergraphs. Theoretical Computer Science. 2012; 463:26-34.
23. Adler Isolde, Georg Gottlob, Martin Grohe. Hypertree width and related hypergraph invariants. European Journal of Combinatorics. 2007; 28(8):2167-2181.
24. Ali Safari Mohammad. D-width, metric embedding, and their connections. Diss. University of British Columbia, 2007.
25. Gottlob Georg, Gianluigi Greco, Francesco Scarcello. Treewidth and hypertree width. Tractability: Practical Approaches to Hard Problems. 2014; 1:20.
26. Marx Dániel. Approximating fractional hypertree width. ACM Transactions on Algorithms (TALG). 2010; 6(2):1-17.
27. Gottlob Georg, *et al*. Hypertree decompositions: Questions and answers. Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, 2016.
28. Gottlob Georg, Zoltán Miklós, Thomas Schwentick. Generalized hypertree decompositions: NP-hardness and tractable variants. Journal of the ACM (JACM). 2009; 56(6):1-32.
29. Chen Hubie, Víctor Dalmau. Beyond hypertree width: Decomposition methods without decompositions. International Conference on Principles and Practice of Constraint Programming. Berlin, Heidelberg: Springer Berlin Heidelberg, 2005.
30. Yolov Nikola. Minor-matching hypertree width. Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms. Society for Industrial and Applied Mathematics, 2018.
31. Liu Tian, Chaoyi Wang, Ke Xu. Large hypertree width for sparse random hypergraphs. Journal of Combinatorial Optimization. 2015; 29(3):531-540.
32. Fujita Takaaki. Relation between ultra matroid and Linear decomposition. Italian Journal of Pure and Applied Mathematics. Accepted, 2015.
33. Fujita Takaaki. Exploring two concepts: Branch decomposition and weak ultrafilter on connectivity system, 2023. arXiv preprint arXiv:2306.14147.
34. Fujita Takaaki. Novel Idea on Edge-Ultrafilter and Edge-Tangle. Asian Research Journal of Mathematics. 2024; 20(4):18-22.
35. Fujita Takaaki. Ultrafilter in Digraph: Directed Tangle and Directed Ultrafilter. Journal of Advances in Mathematics and Computer Science. 2024; 39(3):37-42.
36. Diestel Reinhard, Sang-il Oum. Tangle-tree duality in abstract separation systems. Advances in Mathematics. 2021; 377:107470.
37. Kurkofka Jan, Max Pitz. Ends, tangles and critical vertex sets. Mathematische Nachrichten. 2019; 292(9):2072-2091.
38. Diestel Reinhard, Sang-il Oum. Tangle-tree duality in abstract separation systems. Advances in Mathematics. 2021; 377:107470.
39. Amini Omid, *et al*. Submodular partition functions. Discrete Mathematics. 2009; 309(20):6000-6008.
40. Fujita Takaaki. Shote note: Revisiting Linear Width: Rethinking the Relationship between Single Ideal and Linear Obstacle, 2023. arXiv preprint arXiv:2305.04740.
41. Nešetřil Jaroslav, *et al*. Linear rankwidth meets stability. Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms. Society for Industrial and Applied Mathematics, 2020.
42. Adler Isolde, Mamadou Moustapha Kanté, O-joung Kwon. Linear rank-width of distance-hereditary graphs I. A polynomial-time algorithm. Algorithmica. 2017; 78:342-377.
43. Adler Isolde, Arthur M Farley, Andrzej Proskurowski. Obstructions for linear rank-width at most 1. Discrete Applied Mathematics. 2014; 168:3-13.
44. Kanté Mamadou Moustapha, O-joung Kwon. Linear rank-width of distance-hereditary graphs II. Vertex-minor obstructions. European Journal of Combinatorics. 2018; 74:110-139.
45. Jeong Jisu, O-joung Kwon, Sang-il Oum. Excluded vertex-minors for graphs of linear rank-width at most k. European Journal of Combinatorics. 2014; 41:242-257.
46. Dabrowski, Konrad K, *et al*. Tree pivot-minors and linear rank-width. Acta Mathematica Universitatis Comenianae. 2019; 88(3):577-583.
47. Courcelle Bruno, Mamadou Moustapha Kanté. Graph

operations characterizing rank-width. Discrete Applied Mathematics. 2009; 157(4):627-640.
48. Fomin Fedor V, Sang-il Oum, Dimitrios M. Thilikos. Rank-width and tree-width of H-minor-free graphs. European Journal of Combinatorics. 2010; 31(7):1617-1628.
49. Courcelle Bruno. Several notions of rank-width for countable graphs. Journal of Combinatorial Theory, Series B. 2017; 123:186-214.
50. Moustapha Kanté, Mamadou, Kwon O. An Upper Bound on the Size of Obstructions for Bounded Linear Rank-Width. arXiv e-prints, 2014. arXiv-1412.
51. Fujita Takaaki. Quasi-Ultrafilter on the Connectivity System: Its Relationship to Branch-Decomposition, 2023.
52. Ferry John Douglass. Ultrafilter Membranes and Ultrafiltration. Chemical reviews. 1936; 18(3):373-455.
53. Kupke Clemens, Alexander Kurz, Dirk Pattinson. Ultrafilter extensions for coalgebras. Algebra and Coalgebra in Computer Science: First International Conference, CALCO 2005, Swansea, UK, September 3-6, 2005. Proceedings 1. Springer Berlin Heidelberg, 2005.
54. Booth David. Ultrafilters on a countable set. Annals of Mathematical Logic. 1970; 2(1):1-24.
55. Comfort William Wistar, Stylianos Negrepontis. The theory of ultrafilters. Vol. 211. Springer Science & Business Media, 2012.
56. Koutras Costas D, *et al*. On weak filters and ultrafilters: Set theory from (and for) knowledge representation. Logic Journal of the IGPL, 2021. jzab030.
57. Saks Victor. Ultrafilter invariants in topological spaces. Transactions of the American Mathematical Society. 1978; 241:79-97.
58. Fujita Takaaki. Matroid's Ultrafilter.
59. Dimitrijevic Radoslav. Proximity and uniform spaces. Faculty of Sciences and Mathematics, University of Niš, Serbia, 2009.
60. Leader SA. On cluster in proximity spaces. Fund. Math. 1959; 47:205-213.
61. Leader Solomon. Metrization of proximity spaces. Proceedings of the American Mathematical Society. 1967; 18(6):1084-1088.
62. Leader S. On duality in proximity spaces. Proceedings of the American Mathematical Society. 1962; 13(4):518-523.
63. Di Maio Giuseppe, Somashekhar Naimpally. "$ D $-proximity spaces. Czechoslovak Mathematical Journal. 1991; 41(2):232-248.
64. Mrówka S. Axiomatic characterization of the family of all clusters in a proximity space. Fundamenta Mathematicae. 1960; 48:123-126. Doi: 10.4064/fm-48-2-123-126
65. Almohammed Raghad, Luay A. AL-Swidi. Focal cluster via σ—Topological proximity space. AIP Conference Proceedings. Vol. 2839. No. 1. AIP Publishing, 2023.
66. Talkany AL, Yiezi Kadham Mahdi, Luay AA Al-swidi. On the ψ_ (t (x))-operator proximity in i-topological proximity space. American Institute of Physics Conference Series. 2022; 2398(1).
67. Al Swidi LA, Raghad Almohammed. New concepts via topological proximity space. AIP Conference Proceedings. Vol. 3097. No. 1. AIP Publishing, 2024.
68. Almohammed Raghad, Jabar LAA. Cluster Outer Points via Proximity Spaces. Iraqi Journal of Science, 2024, 863-871.