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Short Survey of Obstruction to Rank-width and Linear Rank Width

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Abstract
Rank-width, as outlined in [15], stands as a prominent graph width parameter in the realm of graph theory. Within this document, we delve into obstructions related to Rank-width. While it may not present groundbreaking novelty, this concise paper serves the purpose of organizing information.

Keywords: Rank Width, Ultrafilter, Tangle, Linear Rank Width

1. Short Introduction

Graph theory is a fundamental branch of mathematics that focuses on the study of networks formed by nodes and edges, examining their paths, structures, and properties. The "graph width parameter" measures the width of a graph, typically representing the maximum width across all cuts or layers in a hierarchical decomposition. Recently, there has been a notable surge in research interest concerning width parameters in both graph theory and algebraic contexts, as evidenced by references [1-13, 22-40, 51]. These width parameters are closely linked to metrics rooted in tree-like structures, commonly referred to as graph decompositions.

In this short survey, we delve into obstructions related to rank-width, a measure of graph complexity using a tree-like decomposition and rank functions of vertex subsets. While it may not present groundbreaking novelty, this concise paper serves the purpose of organizing and consolidating existing information.

2. Preperation: Notations and Definitions

In this section, we explain about notations and definitions in this paper. $V(G)$ represents the set of vertices (nodes) in a graph G, $E(G)$ represents the set of edges in the same graph G, and $G=(V,E)$ signifies that G is a graph defined by a pair of sets, V for vertices and E for edges. Additionally, in this short paper, we utilize the natural number *k, i, and j*.

In this short paper, we use expressions like $A \subseteq X$ to indicate that A is a subset of X, $A \cup B$ to represent the union of two subsets A and B, both of which are subsets of X, or $A = \emptyset$ to signify an empty set. Specifically, $A \cap B$ denotes the intersection of subsets A and B. A similar logic applies to $A \setminus B$. The powerset of a set A, denoted as 2^A, is the set of all possible subsets of A, including the empty set and A itself.

And we provide an explanation of Filters in Boolean Algebras. The definition of a filter in a Boolean algebra $(X, \cup, \cap)$ is given below. We extend the definition of Filters in a Boolean algebra $(X, \cup, \cap)$ to establish a dual relationship with rank-width. **Definition 1:** In a Boolean algebra $(X, \cup, \cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.

(FB1) $A, B \in F \Rightarrow A \cap B \in F$,

(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,

(FB3) $\emptyset$ is not belong to F.

In a Boolean algebras $(X, \cup, \cap)$, a maximal filter is called an ultrafilter and satisfies the following axiom (FB4): (FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

Ultrafilter is a fundamental mathematical concept that has been extensively studied in various fields [52-58]. It is fascinating that seemingly unrelated concepts become connected to rank-width and linear-rank-width when the conditions of rank functions, discussed later, are applied.

3. Obstruction to Rank-width

We provide an explanation concerning the concepts of Rank-width and its obstructions.

Rank-width, as outlined in [15], stands as a prominent graph width parameter in the realm of graph theory. It is characterized by the smallest integer, denoted as k, which allows a graph to be subdivided into a tree-like structure through a recursive partitioning of its vertex set, ensuring that each division results in a matrix with a rank no greater than k. Notably, Rank-width shares a close connection with clique-width [17]. The substantial volume of research devoted to Rank-width, as evident from publications encompassing the range [14-21, 41-50], underscores its significance in the field.

Given that rank-width is a special case of branch-width associated with a symmetric submodular function, we can leverage the theory developed for symmetric submodular functions. Specifically, we can employ concepts known as a "tangle" and an “Ultrafilter” to ascertain the presence of a substantial rank-width in a graph.

To begin, let's delve into the concept of the cut-rank function. The definition of the cut-rank function is provided below.

Definition 2 [15, 17]: For a graph G and a subset A of the vertex set $V(G)$, we can define $\rho(A)$ as the rank of a $|A| \times |V(G) \setminus A|$ 0-1 matrix, denoted as X_A, defined over the binary field. The entry of X_A in the i-th row and j-th column is set to 1 if and only if the i-th vertex in A is adjacent to the j-th vertex in $V(G) \setminus A$.

The cut-rank function ρ exhibits the following inequalities. Note that cut-rank function possesses symmetry and submodularity.

(1) $\rho(A) = \rho(V(G)\setminus A)$ for all $A \subseteq V(G)$,

(2) $\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$ for all $A, B \subseteq V(G)$.

By utilizing the aforementioned cut-rank function, it becomes possible to define a Tangle as follows. Using a similar approach, an Ultrafilter with a deep connection to the Tangle can also be defined.

Definition 3 [17]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a ρ-tangle of order $k+1$ is a set T of subsets of $V(G)$ satisfying the following three axioms.

(T1) For $A \subseteq V(G)$, if $\rho(A) \leq k$, then either $A \in T$ or $V(G) \setminus A \in T$.

(T2) If $A, B, C \in T$, then $A \cup B \cup C \neq V(G)$.

(T3) For all $v \in V(G)$, $V(G) \setminus \{v\} \notin T$.

Definition 4 [13]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a set S of subsets of $V(G)$ is called an order $k + 1$ ρ-ultrafilter if it satisfies the following conditions.

(F1) For any $A, B \in S$ and $\rho(A \cap B) \leq k$, then $A \cap B \in S$.

(F2) For any $A \in S$ and $A \subset B \subseteq V(G)$, if $\rho(B) \leq k$, then $B \in S$.

(F3) $\emptyset$ is not belong to S.

(F4) For any $A \subseteq V(G)$, if $\rho(A) \leq k$, either A ∈ S or $(V(G) / A) \in S$.

Based on previous research results in branch-width, it is possible to derive the following duality theorem.

Theorem 5 [11, 13]: Let ρ be a cut-rank function, and let k be a natural number. For an integer k, a graph G has a ρ-tangle (ρ-ultrafilter) of order k if and only if its rank-width is at least k.

4. Obstruction to Linear-Rank-width

We provide an explanation of the concepts related to Linear-Rank-width and its associated obstructions. Numerous articles within the field of parameter studies address linear width parameters, with a particular focus on width parameters constrained to caterpillars. Similarly, just like Rank-width, there exists a linear version of the graph width parameter known as Linear rank-width.

Linear rank-width is essentially a linear variant, often referred to as the caterpillar version, of rank-width [15, 17]. Just as research on rank-width holds significance, the study of Linear-Rank-width is equally important.It allows for the definition of an obstruction called "ρ-obstacle", “ρ-linear-tangle”, and “ρ-single-ultrafilter” which is outlined below.

Definition 6 [6, 17]: Let ρ be a cut-rank function, and let k be a natural number. For an integer k, ρ-obstacle of order $k + 1$ is a set O of subsets of $V(G)$ satisfying the following three axioms.

(O1) For all $A \in O$, $\rho(A) \leq k$.

(O2) If $A \subseteq B$, $B \in O$ and $\rho(A) \leq k$, then $A \in O$.

(O3) If $A \cup B \cup C = V(G)$, $A \cap B = \emptyset$, $\rho(A) \leq k$, $\rho(B) \leq k$ and $|C| \leq 1$, then exactly one of A and B is in O.

Definition 7 [7]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a set S of subsets of $V(G)$ is called an order $k + 1$ ρ-single-ultrafilter if it satisfies the following conditions.

(S1) For any $A \in S$, $v \in V(G)$, if $\rho(\{v\}) \leq k$ and $\rho(A \cap (V(G)/\{v\})) \leq k$, then $A \cap (V(G)/\{v\}) \in S$.

(S2) For any $A \in S$ and $A \subset B \subseteq V(G)$, if $\rho(B) \leq k$, then $B \in S$.

(S3) $\emptyset$ is not belong to S.

(S4) For any $A \subseteq V(G)$, if $\rho(A) \leq k$, either $A \in S$ or $(X/A) \in S$.

Definition 8 [10]: Let ρ be a cut-rank function, and let k be a natural number. For a graph G, a ρ-linear-tangle of order $k+1$ is a set L of subsets of $V(G)$ satisfying the following three axioms.

(L1) For $A \subseteq V(G)$, if $\rho(A) \leq k$, then either $A \in L$ or $V(G) \setminus A \in L$.

(L2) If $A, B \in L$, $\rho(\{v\}) \leq k$, then $A \cup B \cup \{v\} \neq V(G)$.

(L3) For all $v \in V(G)$, $V(G) \setminus \{v\} \notin L$.

Based on previous research results in linear-branch-width, it is possible to derive the following duality theorem.

Theorem 9 [7, 10, 17]**:** Let ρ be a cut-rank function, and let k be a natural number. A graph G has a ρ-obstacle (ρ-linear-tangle, ρ-single-ultrafilter) of order k if and only if its linear rank-width is at least k.

5. Future tasks: Obstruction to Hypertree-width

We are also very interested in the notion of hypertree width [23] of hypergraphs. Like other graph width parameters, research is actively progressing in the field of hypertree width. Similar to rank-width, there has been a wealth of research results in the field of hypertree width, as evidenced by publications [22-31].

An concept known to have a deep connection with hypertree width is the Hypertangle number. Based on previous research results, it wouldn't be surprising if there were a concept similar to the HyperUltrafilter number, just as the Hypertangle number exists. We plan to continue our investigation in this direction.

6. Future tasks: A proximity and a cluster on a proximity spaces

In the future, we will discuss about relationship between a proximity / a cluster / a semi-ultrafilter on a proximity spaces and rank-width / linear rank-width.

Proximity spaces are a generalization of topological spaces where the concept of "closeness" or "nearness" between sets is defined. They consist of a set along with a proximity relation that satisfies specific axioms, allowing the study of continuity and convergence without relying on a metric (cf. [59-68]).

The definition of a proximity and a cluster and on proximity spaces is provided below.

Definition 10 [59]**:** Let X be a nonempty set. A relation δ on the family $P \in 2^X$ of all subsets of a set X is called a proximity on X if δ satisfies the following conditions:

(P1) If $A\delta B$, then $B\delta A$;

(P2) $A\delta(B \cup C)$ if and only if $A\delta B$, or $A\delta C$

(P3) $X \bar{\delta} \emptyset$;

(P4) $\{x\}\delta\{x\}$ for each $x \in X$

(P5) If $A\bar{\delta}B$, then there exists $E \in P$ such that $A \bar{\delta} E$ and $X \setminus E \bar{\delta} B$

The pair (X, δ) is called a proximity space. We should use the notation $(A, B) \in \delta$ or $(A, B) \notin \delta$ when the sets A and B are either near each other or not, but we shall simply write $A\delta B$ or $A\bar{\delta}B$,

Definition 11 [60]**:** A nonempty family of nonempty subsets of X denoted by C is called a cluster in proximity space (X, δ) if σ satisfies the following conditions:

(C1)For all $A, B \in C$, $A\delta B$;

(C2)$(A \cup B) \in C$ if and only if $A \in C$ or $B \in C$;

(C3)$A\delta B$ for each $B \in C$ implies $A \in C$.

The definition of a semi-ultrafilter and on proximity spaces is provided below.

Definition 12 [65]**:** A nonempty family (cluster) of nonempty subsets of X denoted by F is called a semi-ultrafilter in proximity space (X, δ) if σ satisfies the following conditions:

(C3) $A\delta B$ for each $B \in F$ implies $A \in F$

(C4) Let p and q both belong to X. If $\{p\}$ and $\{q\}$ belong simultaneously to the cluster F, then $p = q$.

(C5) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,

(C6) A is a subset of X and A meets each B from F, *then A also belongs to F.*

We will examine the relationships with rank-width and branch-width by incorporating the conditions of rank functions and symmetric submodular functions into the above concepts.

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8. Conflict of Interest Statement

The author declares no conflicts of interest.

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