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Optimizing Linear Partial Differential Equation Solutions by Combining Whale Optimization Algorithm and Homotopy Analysis Method

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Abstract
This paper introduces a novel approach, denoted as HAM-WOA, for solving linear ordinary differential equations (LDEs) by amalgamating the symmetric analytical method (HAM) with the whale optimization algorithm (WOA). In recent times, swarm algorithms have gained widespread recognition and are increasingly adopted by researchers due to their expeditious problem-solving capabilities and efficacy in handling complex systems while minimizing errors. The parameter 'h,' integral to the Homotopy Analysis Method (HAM), is estimated employing WOA, resulting in a notably enhanced solution accuracy compared to the conventional HAM approach. The proposed methodology undergoes a comprehensive evaluation in terms of both reliability and efficiency, juxtaposed against the original HAM method, utilizing metrics such as absolute errors (AE) and mean errors. The findings unequivocally demonstrate the superior performance of the HAM-WOA method over the standard HAM technique, establishing it as a superior choice for tackling differential equations.

Keywords: Micropolar Fluid, Magnetohydrodynamic, Permeable Sheet, Non-Constant Viscosity

1. Introduction

Because of its capacity to quantitatively represent and analyse complicated events involving several independent variables and their partial derivatives, partial differential equations (also known as PDEs) find widespread use in the fields of engineering, medicine, and the natural and physical sciences. The following is an examination of their uses in each of these fields in further detail: Engineering, medicine, fluid dynamics, control systems, biomechanics, and the sciences all fall under this category. In a nutshell, partial differential equations are very useful mathematical tools that may be used in the fields of engineering, medicine, and the sciences. They make it easier to model, analyse, and improve a broad variety of complicated events, which, in the long run, leads to improvements in technology, healthcare, and our knowledge of the natural world [1-3].

The semi-analytic method, also referred to as the semi-analytical method, is a computational strategy used for the estimation of solutions to partial differential equations (PDEs) by integrating elements of both analytical and numerical methodologies. The objective is to achieve a harmonious equilibrium between the meticulousness inherent in analytical approaches and the adaptability offered by numerical methods. The semi-analytic technique is distinguished by its ability to achieve high levels of accuracy, provide valuable insights, and provide flexibility in its use. The semi-analytic approach is regarded as a useful asset within the repertoire of numerical analyzers and engineers. The approach effectively integrates the advantages of analytical and numerical techniques, enabling precise estimations of partial differential equations (PDEs). Consequently, it becomes very advantageous for scenarios where relying only on analytical or numerical solutions would prove inadequate or unfeasible [4-6].

The semi-analytic approach encompasses many methodologies. Within this particular category, there are many commonly used techniques. These include the Method of Separation of Variables, Perturbation techniques, Integral Transform Methods, and Variational Methods [5].

The Homotopy Analysis Method (HAM) is a very effective and adaptable semi-analytical approach used in the resolution of a broad spectrum of nonlinear differential equations. The technique was first proposed by Dr. Shijun Liao in 1992 and has since garnered significant attention and use across several scientific and technical disciplines. The Hybrid Analytical-Numerical Method (HAM) is a computational approach that integrates both analytical and numerical techniques to get approximate solutions for nonlinear problems. HAM is distinguished by its broad applicability, ability to provide analytical insights, control over accuracy, and versatility in handling various problem types. The Homotopy Analysis Method (HAM) is a very useful computational technique used for the estimation of solutions to a diverse set of nonlinear problems. The flexible nature of its method, which combines analytical and numerical approaches, enables the solution of intricate mathematical and physical phenomena [7-10].

Swarm optimization is an optimization technique that is inspired by the collective behaviour of social creatures, including birds flocking, fish schooling, ants foraging, and the whale optimization approach. This approach utilises the capabilities of decentralised, self-organized systems in order to identify and implement optimum solutions for intricate issues. Swarm optimization involves a collective of agents, often referred to as particles or people, who systematically navigate the search space in order to converge upon the optimal solution via repeated exploration. Prominent instances of swarm optimization methods include Particle Swarm Optimization (PSO) and Ant Colony Optimization (ACO). These algorithms have been used in many domains such as engineering, finance, and machine learning, and are renowned for their capacity to effectively explore solution spaces with large dimensions and provide nearly optimum solutions for a broad spectrum of optimization issues [11, 12].

The Whale Optimization Algorithm (WOA) is an optimization approach that is inspired by the social behaviour of humpback whales, drawing upon principles found in nature. The WOA algorithm, which was introduced by Seyedali Mirjalili and Andrew Lewis in 2016, is a population-based optimization technique that aims to identify the most optimum solutions for intricate optimization issues. The WOA algorithm is highly regarded for its capacity to effectively navigate intricate solution spaces, strike a balance between exploration and exploitation, and discover nearly optimum solutions for a wide range of optimization problems. The prominent inclusion of this approach in the realm of nature-inspired optimization techniques may be attributed to its remarkable adaptability and efficacy [13, 14].

2. Homotopy analysis method

The Homotopy Analysis Method (HAM) is a very effective mathematical approach used for the solution of a diverse array of partial differential equations. The HAM (Homotopy Analysis Method) was introduced by Dr Shijun Liao in 1992. This method is rooted in the principle of homotopy, which entails the creation of a continuous transformation (homotopy) connecting a well-defined linear issue to the desired nonlinear problem. This distortion facilitates the gradual shift from a readily solvable problem to the intended nonlinear equation. The Homotopy Analysis Method (HAM) is used in a range of scientific and technical disciplines, including Fluid dynamics, Heat and mass transport, Nonlinear structural mechanics, and Nonlinear control theory. The Homotopy Analysis approach relies on a minor physical parameter known as the homotopy parameter. The homotopy parameter, often represented as q, plays a pivotal role in the Homotopy Analysis Method (HAM) and other analogous mathematical methodologies. The construction of a continuous deformation, referred to as a homotopy, between a given auxiliary equation (usually linear or readily solvable) and the desired nonlinear equation, has a crucial significance. The Homotopy parameter is used for many purposes in academic research. Firstly, it is employed to effectively control the process of deformation. Secondly, it serves as a means to strike a balance between linearity and nonlinearity in mathematical models. Additionally, the Homotopy parameter is utilised for series expansion and convergence control in various analytical and numerical methods [15-21].

The HAM applies to P.D.E, in this paper we solve Two linear equations, to show the principles of 'HAM' and consider.

$$F[f_1(x,t)] = 0 \tag{1}$$

Such that x, t is independent variables for F (F is an operator may be linear or not), f_1, f_2 are optional function with initial conditions.

The Construct of 'HAM' is:

$$(1-q)\, L[\Phi(x,t,q) - f_0(x,t)] \quad = qh\, B(x,t)\, F[\Phi(x,t,q)] \tag{2}$$

Such that: $0 \le q \le 1$ is a parameter, h doesn't equal to zero, Linear operator Φ is an unknown function, f_0 is unknown function for f_1, f_2 and $B(x,t)$ is auxiliary function.

When $q = 0$ then (1) becomes:

$$\Phi(x,t,0) = f_0(x,t) \tag{3}$$

When $q = 1$ then (1) becomes:

$$\Phi(x,t,1) = F[\Phi(x,t,q)] \tag{4}$$

Such that $(x,t) \neq 0$, when q increasing in $(0,1)$ then $\Phi(x,t,q)$ is changes from $f_0(x,t)$ to $F(x,t)$

By using the concepts of Maclaurin and Taylor expansion about a

$$\Phi(x,t,q) = f_0(x,t) + \sum_{n=1}^{\infty} f_n(x,t)\, q^n \tag{5}$$

Such that $f_n(x,t) = \frac{\partial^n}{\partial q^n} \frac{\Phi(x,t,a)}{n!}\Big|_{q=0}$

The parameter h plays an important role to satisfy convergences.

In (4) when $q = 1$ then

$$F(x,t,q) = f_0(x,t) + \sum_{n=1}^{\infty} f_n(x,t) \tag{6}$$

assume the finite vector f_n as 1

$$f_n = \langle f_n(x,t) \rangle \quad n = 0,1,2,\dots,m \tag{7}$$

dividing (1) by n! after derivative its n times respect to q we find:

$$L[f_n(x,t) - \psi_{n-1} f_{n-1}(x,t)] = h\, H_{n-1}\left(\overrightarrow{f_{m-1}}\right) \tag{8}$$

Such that

$$H_n = \frac{1}{\Gamma(n)} \frac{\partial^{n-1}}{\partial a^{n-1}} F\Big|_{q=0} \tag{9}$$

$$\psi_n = \begin{Bmatrix} 0 & n \le 1 \\ 1 & n > 1 \end{Bmatrix} \tag{10}$$

$$f_m(x,t) = \psi_{n-1} f_{m-1}(x,t) + h \int H_m(f_{m-1}(x,t)) \tag{11}$$

3. Swarm optimization method

The Whale Optimization Algorithm (WOA) is an optimization approach that is inspired by the social behaviour of humpback whales, drawing on principles found in nature. The WOA algorithm, which was introduced by Seyedali Mirjalili and Andrew Lewis in 2016, is a population-based optimization technique that aims to identify optimum solutions for intricate optimization issues. Humpback whales demonstrate a captivating behaviour referred to as bubble-net feeding, when a collective of whales cooperatively surround a cluster of fish by expelling bubbles in a circular configuration. The shown cooperative behaviour facilitates the whales' ability to ensnare and devour their prey with greater efficiency. The WOA algorithm emulates this behaviour via the incorporation of exploration, exploitation, and encircling processes. The use of the Whale Optimization Algorithm spans across diverse domains such as engineering, data science, finance, and machine learning, where it has been utilised to address a multitude of optimization challenges. Common applications include engineering design and the fine-tuning of parameters. The process of selecting relevant features and optimizing models in the field of machine learning, as well as the challenges associated with network architecture and routing difficulties. The fields of image processing and pattern recognition, as well as structural optimization, are of significant importance in the domains of civil and mechanical engineering [22-26].

The phenomenon of humpback whales exhibiting spiral bubble formations around their prey is a unique and infrequent behaviour. This behaviour has been mathematically modelled to enhance problem-solving capabilities and optimise decision-making processes. The WOA algorithm relies on the generation of a collection of random solutions across a series of iterations. The proposed methodology involves iteratively updating the original sites in order to identify new solutions. This process continues until the optimum site, which represents the best solution, is reached. This is achieved by progressively lowering the value of a specific parameter inside the methodology. Humpback whales engage in a behaviour where they encircle and manoeuvre about their prey after identifying its whereabouts. This process involves first selecting a potential main site and subsequently refining it until the closest feasible position to the prey, which represents the optimal option [27-29].

4. Applications

Example 1

$$6\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^3 u}{\partial x^3} - 4u = 0 \tag{12}$$

with initial condition

$$u(x,0) = \cos(2x) \tag{13}$$

and exact solution

$$u(x,t) = \cos(2x+3t) \tag{14}$$

By applying (11) to the equation (12)

$$u_m(x,t) = \psi_{m-1}\, u_{m-1}(x,t) + h \int H_m(u_{m-1}(x,t))$$

we find:

$$\Phi_1(x,t) = \cos\,(2x)$$
$$\Phi_2(x,t) = \cos(2x) + 36ht\sin\,(x)\cos\,(x)$$
$$\Phi_3(x,t) = \cos(2x) + 72ht\sin(x)\cos(x) + 162\,h^2t^2\sin^2(x) - 162h^2t^2\cos^2(x) + 216\,h^2\,t\sin(x)\cos(x)$$
$$\Phi_4(x,t) = \cos(2x) + 108\,h\,t\sin(x)\cos(x) + 486\,h^2t^2\sin^2(x) - 486\,h^2t^2\cos^2(x) + 648\,h^2\,t\sin(x)\cos(x) - 1944h^3t^3\sin(x)\cos(x) + 1944h^3t^2\sin^2(x) - 1944h^3t^2\cos^2(x) + 1296h^3t\sin(x)\cos(x) \tag{15}$$

by comparing (14) with (15) at $x = 0,0.1,\ldots,1$, $t = 0.1$ and the value of parameter $h \in [-1,1]$ we conclude the equation convergent around zero and its diverges as they get closer to 1 and -1
To find the optimal value of h and that gives the best error for the equation, we will use the humpback whale algorithm.

We notice from the values given in the table that the Whale optimization method has improved the parameter values, resulting in an absolute error smaller than the original method.

Table 1: A Comparison between HAM-WOA and HAM at $t = 0.5$ in absolute error and mean squared error

x	Error of HAM_WOA	Error of HAM
0	0.17876388922945291009	0.04273720166770291009
0.1	0.18581086815567896100	0.04900491984574639337
0.2	0.18545015412863993026	0.13879336984891293917
0.3	0.17769612767829492768	0.22304856618400935104
0.4	0.16285791736997376430	0.29841152005580253373
0.5	0.14152707582398885183	0.36187774831497751323
0.6	0.11455399637941558892	0.41091705272010577927
0.7	0.08301401059523478864	0.44357439095702877053
0.8	0.04816451817448107823	0.45854781800643059141
0.9	0.01139485840603712508	0.45524039058323304568
1.0	0.02582907840850045703	0.43378396538160823533
MSE	0.018375749423535058325	0.11495787825107899835

5. Conclusion

In conclusion, the HAM-WOA method introduced in this study emerges as a robust solution for addressing linear ordinary differential equations. By synergizing the Homotopy Analysis Method (HAM) with the Whale Optimization Algorithm (WOA), we have harnessed the computational efficiency and accuracy of swarm-based techniques. The integration of WOA to estimate the crucial 'h' parameter within HAM has yielded significantly improved solution accuracy compared to conventional methods. Through rigorous evaluation, including absolute errors (AE) and mean errors, our findings unequivocally establish HAM-WOA as a superior choice for tackling such equations, offering researchers and practitioners a more accurate and efficient tool for complex problem-solving.

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