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On the Crossing Numbers of Corona Product of Planar Graph G with K_n

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Abstract
The crossing numbers of graphs is the least number of edge crossings in all possible good drawing of graph G. Corona Product of graphs has many interesting graph theoretical properties. In this paper, we analyse the crossing numbers of Corona Product of Planar Graph G with K_n. We have proved $Cr(G \text{ o } K_n) \leq m\, Z(n+1)$ and equality holds for $4 \leq n \leq 11$.

Keywords: Crossing Number, Corona Product of Graphs, Planar Graph, Path, Cycle

MSC Code (2010): 68R10; 05C10; 05C62

1. Introduction
The Crossing Number Cr (G) of graph G is the least number of edge crossings among the drawings of G in the plane. A good drawing of a graph G satisfies the following: i) adjacent edges never cross, ii) two non-adjacent edges cross at most once, iii) no more than two edges cross at a point of the plane, iv) no edge passes through a vertex of graph G. A drawing of a graph is good if and only if all edges intersect at most once.
The corona product of two graphs was defined by Frucht and Harary in [2]. In this paper we have evaluated the crossing numbers of Corona product of cycle C_m with a complete graph K_n and crossing numbers of corona product of any planar graph G of order m with a complete graph K_n for $4 \leq n \leq 11$ and upper bound for any natural number $n \geq 12$.

2. Crossing Number of Corona Product of Planar Graph G with Complete graph K_n
Definition 2.1: (Corona Product of graphs) Let G and H are the two graphs then the corona product of G and H is denoted by GoH and it contains one copy of G, called the centre graph, | V (G) | copies of H, called the outer graph, and making i^{th} vertex of G adjacent to every vertex of i^{th} copy of H, where $1 \leq i \leq |V(G)|$.
Definition 2.2: (*Planar Graph*) A graph is said to be a planar graph or embeddable in the plane if crossing numbers of the graph are zero.

Lemma 2.1: Let A, B, C are mutually disjoint subsets of E. Then

$$Cr_D(A \cup B) = Cr_D(A) + Cr_D(B) + Cr_D(A, B)$$
$$Cr_D(A, B \cup C) = Cr_D(A, B) + Cr_D(A, C)$$

Where D is a good drawing of G.

Lemma 2.2: Let $Z(n,m) = \left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor \left\lfloor\frac{m}{2}\right\rfloor \left\lfloor\frac{m-1}{2}\right\rfloor$

1. $Cr(K_{n,m}) = Z(n, m)$ where $\min\{m, n\} \leq 6$.
2. $Cr(K_{n,m}) \leq Z(n, m)$ for $m, n \in N$.

Remark 2.1: If G be a planar graph on m vertices, then $Cr(GoK_n) = 0$ for $n \leq 3$.

Theorem 2.1: If G be any planar graph with m number of vertices, then $Cr(GoK_4) = m$.

Proof: Let G be a planar graph with m vertices. Let $V(G) = \{a_1, a_2, \cdots, a_m\}$. We know that a planar graph partitions the plane into several regions. These regions may be interior or exterior. By definition of Corona product, we have to place m copies of a good drawing of K_4 i.e K_4^i for $1 \leq i \leq m$ either in the interior region which is closed to vertex a_i for $1 \leq i \leq m$ of G or in the exterior region and joining each vertex a_i of graph G with every vertex of ${K_4}^i$ for $1 \leq i \leq m$, where K_4^i is the i[th] copy of K_4 as shown in Fig 1.

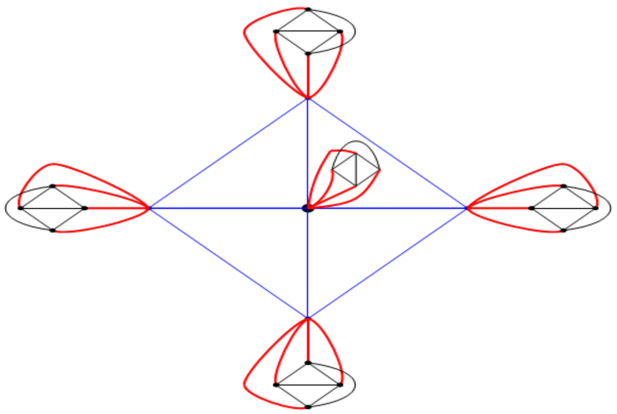

Fig 1: (GoK_4)

Join of K_4^i with a vertex a_i of G gives a drawing isomorphic to K_5. We know that $Cr(K_5) = 1$. Thus $Cr(K_4^i + \{a_i\}) = 1$ for $1 \leq i \leq m$. Since $E(GoK_4) = E(G) \cup E\left[\bigcup_{i=1}^{m} K_4^i + \{a_i\}\right]$. Since G is planar thus we get,

$$Cr(GoK_4) \leq \sum_{i=1}^{m} Cr({K_4}^i + \{a_i\}) \leq \sum_{i=1}^{m} .1$$

On the otherhand any good drawing of GoK_4 has at least m nonintersecting copies of K_5.

$\therefore Cr(GoK_4) \geq m$

Hence proved.

Theorem 2.2: If G is any planar graph with m number of vertices, then $Cr(GoK_n) = m\,Cr(K_{n+1})$ for $n \geq 4$.

Proof: Let G be a planar graph such that $|V(G)| = m$. Firstly we fixed n and prove the result by the method of induction on m. The result holds for m=1, because $Cr(GoK_n) = Cr(K_1 + K_n) = Cr(K_{n+1})$.

Let us assume the result holds for all subgraph H of G such that $|V(H)| = m - 1$

i.e $Cr(HoK_n) = (m - 1)Cr(K_{n+1})$

Now we have to prove it for m.

Let if possible $Cr(GoK_n) = mCr(K_{n+1}) - 1$. By removing all edges of one copy of K_{n+1} from the drawing of GoK_n we get a drawing of HoK_n with crossings $mCr(K_{n+1})-1-Cr(K_{n+1}) = (m-1)Cr(K_{n+1})-1$, which is a contradiction for induction hypothesis. Thus, by induction the result holds for m.

Now we have to fix m and prove the result by using induction on n.

The result holds for n=4 by theorem 2.1. Assume the result hold for n-1, i.e

$Cr(C_m o K_{n-1}) = mCr(K_n)$

Let if possible $Cr(C_m o K_n) = mCr(K_{n+1}) - 1$. If we remove a vertex say b_i, $1 \leq i \leq m$ which is not a vertex of G from each copy of K_{n+1} of drawing GoK_n we get drawing of GoK_{n-1} such that,

$$\begin{aligned}
Cr(GoK_{n-1}) &\leq mCr(K_{n+1}) - 1 - \sum_{i=1}^{m} Cr(b_i) \\
&\leq \left[\sum_{i=1}^{m} Cr(K_{n+1}) - \sum_{i=1}^{m} Cr(b_i)\right] - 1 \\
&\leq \sum_{i=1}^{m} [Cr(K_{n+1}) - Cr(b_i)] - 1 \\
&\leq \sum_{i=1}^{m} [Cr(K_n)] - 1 \\
&\leq m[Cr(K_n)] - 1
\end{aligned}$$

Which is a contradiction for induction hypothesis. By induction the result holds for all $n \geq 4$.

Corollary 2.1. If G be any planar graph with m vertices then $Cr(GoK_n) \leq mZ(n + 1)$ for any $n \geq 4$ and equality holds for $4 \leq n \leq 11$.

Proof. Proof followed by theorem 2.2 and lemma 2.1.

Conclusion

In this paper we obtained the exact crossing number of Corona products of any planar graph with K_n.

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