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Condition of Unique Solutions on the Size Spaces

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Abstract
Proving the uniqueness of the solutions after obtaining them requires complex conditions, especially in the size spaces. This paper proves the equivalence of Cauchy sequences that contains the solution's existence in the sizes spaces, as a prerequisite for proving the uniqueness of the resulting solutions.

Keywords: Normed Space, Equivalent, Fixed Point

MSC 2000: 46B40, 40D25, 37J10

1. Introduction and Preliminary

Studies of the existence of the solution to the differential, integral, or boundary equations in various sciences is a fertile field in which researchers present their results continuously and often use fixed-point theorems to prove the existence of the solution, see [2], [5-7], [12-15]. This does not necessarily mean proving the uniqueness of the solution.

Sizes spaces are one of the spaces that are avoided by authors as a result of their difficulty and breadth. It is difficult to identify a point in sized spaces and then prove that this point is unique. But if we can prove the equivalence of the sequences, then we can prove the uniqueness of the solution based on this paper. H. Gunawan, [10] established the idea of sizes spaces by introducing the definition of n-normed space. Recently, Raj *et al*. [16] proved some properties in such space.

Definition 1.1. [16] A sequence u_k in an n-normed spaces $(U, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k\to\infty} \|u_k, v_2, \dots, v_n\| = 0, \tag{1.1}$$

for all $z_2, \dots, z_n \in U.$

Recent papers that discuss of the sizes spaces and their characteristics, we refer to [1], [3, 4], [8, 9], [11].

The main result of this paper is to explore the conditions that make possible the uniqueness of solutions in sized spaces.

2. Main Results

Definition 2.1. n-Cauchy sequences $\{u_{1_k}\}, \{u_{2_k}\}, \dots, \{u_{n_k}\}$ in an linear n-normed spaces $(U, \|\cdot, \dots, \cdot\|)$ are said to be equivalent, denoted by $\{u_{1_k}\} \simeq \{u_{2_k}\} \simeq, \dots, \simeq \{u_{n_k}\}$ if for every neighborhood H of 0 there is an integer $N(H)$ such that $1_k, \dots, n_k \geq N(H)$ implies:

$$u_{1_k} - u_{2_k} - \dots - u_{n_k} \in H \Rightarrow \|u_{1_k} - u_{2_k} - \dots - u_{n_k}, z_2, \dots, z_n\| \in H, \tag{2.1}$$

with respect to the independent set $\{z_2, \dots, z_n\}$ in U.

Theorem 2.1. For every $z_2, \dots, z_n \in U$, $\{u_{1_k}\} \simeq \{u_{2_k}\} \simeq, \dots, \simeq \{u_{n_k}\} \in (U, \|\cdot, \dots, \cdot\|)$ if and only if

$$\lim_{k\to\infty} \|u_{1_k} - u_{2_k} - \dots - u_{n_k}, z_2, \dots, z_n\| = 0.$$

Proof

Let $\{u_{1_k}\} \simeq \{u_{2_k}\} \simeq, \dots, \simeq \{u_{n_k}\}$ then for every neighborhood H of 0 there is an integer $N(H)$ such that $1_k, \dots, n_k \geq N(H)$ implies that (2.1) is satisfies for every $z_2, \dots, z_n \in U$, using Definition 1.2. to getting $\lim_{k\to\infty} \|u_{1_k} - u_{2_k} - \dots - u_{n_k}, z_2, \dots, z_n\| = 0$ which is proof the part if. To proof the part only if: let $\lim_{k\to\infty} \|u_{1_k} - u_{2_k} - \dots - u_{n_k}, z_2, \dots, z_n\| = 0$, concluding $u_{1_k} \simeq u_{2_k} \simeq, \dots, \simeq u_{n_k}$ are n-Cauchy sequences in $(U, \|\cdot, \dots, \cdot\|)$, then $\|u_{1_k} - u_{2_k} - \dots - u_{n_k}, z_2, \dots, z_n\| \in H$ for every $z_2, \dots, z_n \in U$, such that H a neighborhood of 0 and $u_{1_k} - u_{2_k} - \dots - u_{n_k} \in H$, when there exist an integer $N(H)$

Hence, $\{u_{1_k}\} \simeq \{u_{2_k}\} \simeq, \dots, \simeq \{u_{n_k}\}$.

Theorem 2.2 If $\{u_{1_k}\}$ is equivalent to $\{a_{1_k}\}$, $\{u_{2_k}\}$ is equivalent to $\{a_{2_k}\}$ and $\{u_{n_k}\}$ is equivalent to $\{a_{n_k}\}$ in $(U, \|\cdot, \dots, \cdot\|)$ then for all $k \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $z_2, \dots, z_n \in U$.

1. $\{u_{1_k} + u_{2_k} + \dots + u_{n_k}\}$ is equivalent to $\{a_{1_k} + a_{2_n} + \dots + a_{n_k}\}$.
2. $\{\gamma u_{1_k}\}$ is equivalent to $\{\gamma a_{1_k}\}$,…, $\{\gamma u_{n_k}\}$ is equivalent to $\{\gamma a_{n_k}\}$.

Proof

$$\begin{aligned}
&\|(u_{1_k} + u_{2_k} + \dots + u_{n_k}) - (a_{1_k} + a_{2_n} + \dots + a_{n_k}), z_2, \dots, z_n\| \\
&= \|(u_{1_k} - a_{1_k}) + (u_{2_k} - a_{2_k}) + \dots + (u_{n_k} - a_{n_k}), z_2, \dots, z_n\| \\
&\leq \|(u_{1_k} - a_{1_k}), z_2, \dots, z_n\| + \|(u_{2_k} - a_{2_k}), z_2, \dots, z_n\| + \dots + \|(u_{n_k} - a_{n_k}), z_2, \dots, z_n\| \\
&= 0,
\end{aligned}$$

When $k \to \infty$. Using Theorem 3.1 to obtain $\{u_{1_k} + u_{2_k} + \dots + u_{n_k}\} \simeq \{a_{1_k} + a_{2_n} + \dots + a_{n_k}\}$. Then (i) is proved. To prove (ii), taking $k \to \infty$, where $z_2, \dots, z_n \in U$.

$$\begin{aligned}
&\|(\gamma u_{1_k} - \gamma a_{1_k})(\gamma u_{2_k} - \gamma a_{2_k}) \dots (\gamma u_{n_k} - \gamma a_{n_k}), z_2, \dots, z_n\| \\
&= |\gamma| \|(u_{1_k} - a_{1_k})(u_{2_k} - a_{2_k}) \dots (u_{n_k} - a_{n_k}), z_2, \dots, z_n\| \\
&= |\alpha|.0
\end{aligned}$$

Hence, $\{\gamma u_{1_k}\} \simeq \{\gamma a_{1_k}\}$, $\{\gamma u_{2_k}\} \simeq \{\gamma a_{2_k}\}$ and $\{\gamma u_{n_k}\} \simeq \{\gamma a_{n_k}\}$.

Theorem 2.3 The relation $\simeq$ on the set of n-Cauchy sequences on U is equivalent relation in $(U, \|\cdot, \dots, \cdot\|)$.

Proof.

i) Since, $\{u_{1_k}\} \simeq \{u_{1_k}\} \simeq \dots \simeq \{u_{1_k}\}$, then the reflexivity property is satisfied.

ii) For any permutation $i_1, i_2, \dots, i_n$, in $\{u_{1_i}\} \simeq \{u_{2_i}\} \simeq, \dots, \simeq \{u_{n_i}\}$ we get that $\{u_{1_k}\} \simeq \{u_{2_k}\} \simeq, \dots, \simeq \{u_{n_k}\}$ then the symmetry and transitive properties have been fulfilled.

Hence, $\simeq$ is the equivalent relation on $(U, \|\cdot, \dots, \cdot\|)$.

3. Paper Significance

This paper presents a study through which we can prove the uniqueness of the solutions in the sizes spaces without the need for complex additional conditionals.

4. Conclusion

Proving the equivalence of Cauchy sequences is important in the size spaces so that we can show that the fixed points are unique using the Banach contraction principle.

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