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Application of Space-Time Localized Radial Basis Functions Scheme to Solve the American and European Options Pricing Models

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Abstract

In this paper, we investigate the use of space-time localized radial basis function collocation method to solve the options pricing European and American models. The application of the proposed method to options pricing European and American models open a new area in the development of this technique for solving partial differential equations (PDEs) with variable coefficients. Beside the known advantages of the method when solving the PDEs with constant coefficients, re-computing the resulting matrix and solving the algebraic system at each time level, as done by time stepping method when solving PDEs with variable coefficients, are avoided. In herein, the numerical approximation of the optimal exercise boundary in the case of American options is obtained effectively by using a new algorithm of penalty iterative scheme. The same technique is applied to the two and three-dimensional American options without boundary conditions when considering the problem in the space domain. Results are compared with analytical and available published numerical results. Obtained results indicate that the proposed scheme offers accurate approximation compared with existing numerical methods applied to such option pricing European and American models.

Keywords: Black-Scholes problem, European and American options, American Multi-Asset Option Pricing, Localized Radial Basis Functions, Space-Time Scheme, Penalty Method

1. Introduction

Solving partial differential equations arising in financial markets is nowday considered as one of the most important problems that attract the attention of both mathematicians and scientists. Many techniques have been developed and applied to solve European and American options. Finite element method, boundary element method and finite differential method are the most famous used mesh methods. All of them are based on seeking the approximate solution at each time level t using time discretization scheme.

Recently, some papers treating the multi-asset American put option were published. Among them, we can mention the work based on a semi-implicit finite element scheme published by R. Zhang *et al*. [5]. They have used a penalty method to transform the linear complementary problem (LCP) defining the multi-asset American put option into a nonlinear parabolic problem on an unbounded domain. For solving the problem, they implemented the far field boundary condition to determinate a rectangular truncated domain after variables change. The mentioned mesh methods show difficulties when it has to be applied to option pricing problem with number of multi-asset great than four. During the last decade and under the intensive research on meshless methods, many papers [6–18] have been published concerning the application of radial basis function approach as meshfree method for solving PDEs in financial option pricing. In many of these works, equations governing European and American options were firstly time discretized by using time integration schemes and then, the obtained option pricing equations in space were solved at each time level by using radial basis function meshless method. To complete the use of such meshless method and because of the use of time integration scheme, time stability analysis issue has to be discussed. In [15] two schemes of a meshless local weak form of the boundary element method (LBEM) for option pricing are presented. The first one is based on the moving least squares approximation (MLS) and the second on Wu's compactly supported radial basis functions. The problem of the free boundary arising in American option is reduced to a problem with an unbounded fixed boundary using a Richardson extrapolation technique. The problem is then solved in a truncated domain $[0, S_{max}]$, where S_{max} is chosen five time of the strike price $S_{max} = 5E$. The θ-method is used for time discretization and the approximate solution is obtained at every time step level by solving a sparse system of linear equations. In [16] J.A. Rad *et al*. had also used the radial basis point interpolation (RBPI) to solve the Black-Scholes model for European and American options. To overcome the problem due to the free boundary that arises in the case of American options, three different approaches are used: the projected successive over relaxation method (PSOR), the Bermudan approximation and the penalty approach. For time discretization the θ -method is adopted, and for the semi-infinite domain consideration an exponential change of variables is implemented to transform the unbounded domain $[0,+\infty)$ into bounded one $[0,1]$. Only single-asset option is treated in this work. In [19] J. Martín Vaquero *et al* applied penalty method given in [16] and the stabilized explicit Runge-Kutta scheme to solve multi-asset American options. The truncated domains are chosen to be $[0,5E]$ and $[0,2.5E]$ for 1-D examples, $[0,4E]\times[0,4E]$ for 2-D example and $[0,4E]\times[0,4E]\times[0,4E]$ for 3-D example. In all these works, they made difference between space and time variables and discussed the time stability analysis.

In this paper the numerical scheme developed by Hamaidi *et al*. in [1–3], called the space-time localized RBF collocation method is applied to solve the Black-Scholes equations governing European and American options. The technique is based on the transformation of the d-dimensional (PDE) into $(d+1)$-dimensional one by combining the d-dimensional vector space variable and 1-dimensional time variable in one $(d+1)$-dimensional variable vector and solving the problem without using any time discretization techniques like implicit, explicit, the method-of-line approach and others as done by classical methods. We can also recall that the applied method generate a sparse and square system matrix which is one of the advantages of the method in respect to other formulations of space-time meshless methods given in [21–24]. Another advantage of the presented space-time localized RBF collocation method in comparison to other, as it was mentioned in [1], is that the time stability analysis discussion is avoided. This advantage is in good concordance with the results mentioned by Hon and Xian [8] which is "*In the case of the European options, it is shown that the major numerical error is from the time integration instead of the spatial approximation by comparing with the analytical solution*". But the main advantage herein is the reduction of computation efforts significantly since no need to re-compute the matrix for the resulting algebraic system at each time level, unlike the case for others time integration methods used to solve PDEs with time-dependent coefficients. Based on the discussion above the considered options can be solved directly without any time steps scheme. For the American options, an algorithm of iterative penalty method is necessary for seeking the approximate solution since the problem can be treated as a linear complementary one. Following, [12, 14, 20], we can also mention that the two and three asset American options problem is treated without any boundary conditions when the problem is considered in space domain.

The remainder of the paper is as follows. In Section 2, we present the formulation of European and American option pricing models and also the penalty formulation of the American option. In Section 3 we recall the space-time localized radial basis function collocation method for solving parabolic PDEs. Numerical results of the European option case and comparison with analytical solution is given in section 4. In the same section 4, the penalty algorithm for American options valuation and results of one, two and three-dimensional put determination and its comparison with the binomial method and some other numerical techniques are also given. Conclusion is presented in the last Section 5. Furthermore, this numerical scheme proposed herein is of a general nature and can be used for solving other higher dimension option pricing problems.

2. European and American Options Formulation

In financial literature, it has been demonstrated that the European and American options with maturity T and strike price E on an underlying asset S follow the stochastic differential equation.

$$dS = S(rdt + \sigma dW) \tag{1}$$

Where r and σ are the risk-free interest rate and the volatility of the stock price S respectively. Under some known assumptions, the option value $V(S,\tau)$ satisfies the one-dimensional Black-Scholes equation defined by

$$\frac{\partial V}{\partial \tau} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} = rV, \qquad \tau \in [0,T[, \qquad S \in \mathbb{R}^+, \tag{2}$$

Before illustrating how to apply the radial basis functions as a spatial-temporal collocation scheme for options pricing, we first give a review of the two options.

2.1 European Options

First, we consider the one-dimensional European options problem, which satisfies the Black-Scholes equation given by (2). To solve numerically this problem, we consider a truncated domain $\Omega = [a,b]\times[0,T]$. The problem has a final condition under the form $V(S,T) = h(S)$ (payoff), and boundary conditions $V(a,\tau) = \alpha(\tau),\ \forall\tau \in [0,T]$ and $V(b,\tau) = \beta(\tau),\ \forall\tau \in [0,T]$. The payoff condition is given by

$$h(S) = V(S,T) = \begin{cases} C(S,T) & = & \max(S-E,0) \text{ for Call option,} \\ P(S,T) & = & \max(E-S,0) \text{ for Put option,} \end{cases} \tag{3}$$

Since the the payoff condition (3) is backward in time, we need to use a new state variable $t = T - \tau$ to transform the condition into an initial one and get a new form of the equation (2) as follows

$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rV = 0, \qquad t \in]0,T], \qquad S \in [a,b] \tag{4}$$

The equation (4) can be solved directly under its form as partial differential equation with variable coefficients or transformed into equation with constant coefficients using the map $x = \ln(S)$. Hence, the new equation is represented under the form

$$\frac{\partial U}{\partial t} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial U}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + rU = 0, \quad t \in]0,T], \quad x \in [\ln(a), \ln(b)] \tag{5}$$

with the following initial and boundary conditions:

$$\begin{cases} U(\ln a, t) &= \alpha(T-t), & \forall t \in]0,T], \\ U(\ln b, t) &= \beta(T-t), & \forall t \in]0,T], \\ U(x,0) &= g(x), & \forall x \in [\ln(a), \ln(b)], \end{cases}$$

Where,

$$g(x) = \begin{cases} \max(e^x - E, 0) & \text{for } Call, \\ \max(E - e^x, 0) & \text{for } Put, \end{cases} \tag{6}$$

The imposed artificial boundary conditions are given by setting α and β to be

$$\begin{cases} \alpha(\tau) = 0, & \beta(\tau) = b, \quad \text{for} \quad Call, \\ \alpha(\tau) = E e^{-r(T-\tau)}, & \beta(\tau) = 0, \quad \text{for} \quad Put, \end{cases}$$

2.2 American Options

2.2.1 American One-Asset Option

In the most published works about the numerical solution of the American options valuation, the problem was treated as a free boundary value problem. In our best knowledge, up to now no analytical formula for such problem is available. The American options allow early exercise at any time $\tau \in [0,T]$ with optimal exercise stock value $S = B(T-\tau)$. The difficulty to compute an accurate approximate solution of the American options using the most numerical methods is due to the unknown free boundary $S = B(T-\tau)$.

Until optimal exercise, we have the following equation

$$\frac{\partial V}{\partial \tau} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0, \qquad S > B(\tau) \tag{7}$$

and at the optimal exercise moment we define the following condition

$$V(S,\tau) = V(S,T) = h(S) \tag{8}$$

In American option, we always have

$$V(S,\tau) \geq V(S,T) = h(S) \tag{9}$$

So, the American options equation to be solved is under the following form:

$$\begin{cases} \frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rV = 0, \quad \forall t \in [0,T], & S > B(T-t), \\ V(S,t) = h(S), & \text{otherwise}. \end{cases} \tag{10}$$

Based on these equations (9) and (10), the American option satisfies the following strong form of the Linear Complementarity Problem (LCP)

$$\begin{cases} LV(S,t) = \frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rV \geq 0, & \forall (S,t) \in \Omega \times [0,T] \\ V(S,t) \geq h(S), & \forall (S,t) \in \Omega \times [0,T], \\ LV(S,t) \times \left(V(S,t) - h(S)\right) = 0 & \forall (S,t) \in \Omega \times [0,T], \end{cases} \tag{11}$$

Various penalty methods were implemented in [5, 6, 19, 25–27] to solve the American options problem under its LCP form. In herein we adopt a new algorithm based on the penalty method given in [6, 27]. For this approach, a simple penalty term is added to the main equations to enforce the early exercise constraint. The solution of the resulting nonlinear equation can be computed via a penalty iteration method that is based on solving a sequence of partial differential equations of the form:

$$LV(S,t) - p \times \left(h(S) - V(S,t)\right)^{+} = 0, \qquad \forall (S,t) \in \Omega \times [0,T] \tag{12}$$

Where $(a)^{+} = \max(a,0)$ and p is a penalty parameter greater than one.

2.2.2 American Multi-Asset Option

Generalizing the American one-Asset option, it is well known that the mathematical model of American multi-asset option can be defined by the partial differential equation:

$$\mathrm{L}V = \frac{\partial V}{\partial t} - \frac{1}{2}\sum_{i=1}^{i=d}\sum_{j=1}^{j=d} \rho_{i,j}\,\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} - \sum_{i=1}^{i=d} r \frac{\partial V}{\partial S_i} + rV.$$

By using the variable change $t = T - \tau$, can be seen as a Linear Complementarity Problem of the form:

$$\begin{cases} \mathrm{L}V(S,t) \geq 0, & \forall (S,t) \in \Omega \times [0,T], \\ V(S,t) = h(S), & \forall (S,t) \in \Omega \times [0,T], \end{cases} \quad \text{or} \quad \begin{cases} \mathrm{L}V(S,t) = 0, & \forall (S,t) \in \Omega \times [0,T], \\ V(S,t) \geq h(S), & \end{cases} \tag{13}$$

Where V is the value of the contract, S_i is the value of the $i^{\,th}$ underlying asset $S = (S_i)_{i=1,d}$, d is the number of underlying assets, ρ_{ij} is the correlation between asset i and asset j, and r is the risk-free interest rate. The function $h(S)$ is the payoff defined by

$$h(S) = \max\left(E - \sum_{i=1}^{i=d} \alpha_i\, S_i, 0\right),$$

Where α_i is the weight of the $i^{\,th}$ asset and the space domain is $\Omega = \prod_{i=1}^{i=d}[\,0,+\infty[$.
Following the same technique applied to American one Asset option, the problem (13) is transformed into a penalty one defined by

$$\mathrm{L}V(S,t) - p \times \left(h(S) - V(S,t)\right)^{+} = 0, \qquad \forall (S,t) \in \Omega \times [0,T].$$

3. Space-Time Localized RBF Method Formulation

To recall the space-time localized RBF method defined in, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a sufficiently regular boundary $\partial\Omega$ and consider the following time-dependent boundary value problem

$$\frac{\partial}{\partial t}V(x,t) + \mathcal{P}_{(x,t)}V(x,t) = f(x,t), \qquad \forall x \in \Omega,\ \forall t \in]0,T], \tag{14}$$

$$\mathcal{B}V(x,t) = g(x,t), \qquad \forall x \in \partial\Omega,\ \forall t \in]0,T[, \tag{15}$$

$$V(x,0) = u_0(x), \qquad \forall x \in \Omega, \tag{16}$$

Where $\mathcal{P}(x,t)$ is a differential operator of second order with variable coefficients representing either European or American pricing model and written under the following general form

$$\mathcal{P}_{(x,t)}V = \sum_{i=1}^{i=d}\sum_{j=1}^{j=d} a(x,t)\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{i=d} b(x,t)\frac{\partial V}{\partial x_i} + c(x,t)V,$$

$\mathcal{B}$ is a boundary operator depending on the kind of boundary condition treated and f, g and u_0 are sufficiently regular data functions.
To solve the problem we first need to transform the d-dimensional time depend problem (14-16) into a $(d+1)$-dimensional boundary value problem in space-time domain $\Omega_T = \Omega \times]0,T[$. Fig 1 represents the space-time domain concept of 1D and 2D domains. The boundary of the new formulated domain Ω_T is given by $\partial\Omega_T = \Omega \times \{t = T\} \cup \partial\Omega \times [0,T] \cup \Omega \times \{t = 0\}$.

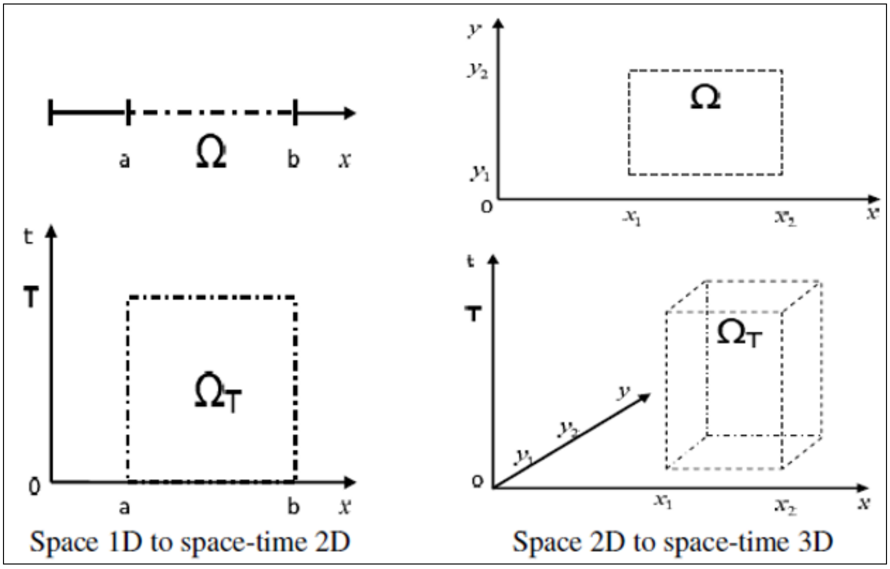

Fig 1: Space-time domain concept

Following the technique discussed in [1], the problem given by Eqs. (14-16) is then formulated by considering the Eq. (14) given by $\frac{\partial}{\partial t}V + \mathcal{P}_{(x,t)}V = f(x,t),\ \forall(x,t) \in \Omega \times]0,T[$ as a domain equation in Ω_T and Eqs. (15) and (16) as boundary conditions on $\partial\Omega \times [0,T]$ and $\Omega \times \{t = 0\}$ respectively. To complete the set of boundary conditions, we also consider the Eq. (14) as a boundary condition on $\Omega \times \{t = T\}$.

The resulted space-time problem is then defined by

$$\begin{aligned} \mathbf{L}_{(x,t)}V(x,t) &= f(x,t), \quad \forall(x,t) \in \Omega_T, \\ \mathbf{B}_{(x,t)}V(x,t) &= h(x,t), \quad \forall(x,t) \in \partial\Omega_T, \end{aligned} \tag{17}$$

Where $\mathbf{L}_{(x,t)} = \frac{\partial}{\partial t} + \mathcal{P}_{(x,t)}$ in Ω_T, $\mathbf{B}_{(x,t)} = [\mathcal{B}_{(x,t)}\ \ \mathbf{1}]$ and $h = [g\ \ u_0]$ on $\partial\Omega_T$.

To solve such problem using the localized RBF method, we first need to drive the local approximation of the unknown function V and then the local approximation of $\mathbf{L}_{(x,t)}V(x,t)$ and $\mathbf{B}_{(x,t)}V(x,t)$ can be determined easily base on the components of the function V. So, the local approximation of V in an influence domain Ω_T^s associated to a selecting collocation point $\hat{x}_s = (x_s, t_s)$ and containing a number n_s of nearest neighboring points $\{\hat{x}_k^{[s]} = \left(x_k^{[s]}, t_k^{[s]}\right)\}_{k=1}^{n_s} \in \Omega_T^s$ (Fig 2), is given by:

$$V(\hat{x}_s) \simeq \hat{V}(\hat{x}_s) = \sum_{k=1}^{n_s} \alpha_k \Phi\left(\|\hat{x}_s - \hat{x}_k^{[s]}\|\right), \tag{18}$$

Where $\{\alpha_k\}_{k=1}^{n_s}$ are the unknown coefficients, $\|\cdot\|$ is the Euclidean norm, and Φ is the chosen RBF. There is many different RBF to be chosen. Among them we can mention the multiquadric function $\phi(r) = \sqrt{(\epsilon r)^2 + 1}$, where ϵ is the shape parameter, that has been proved, in many references, to be the most effective one these last decade.

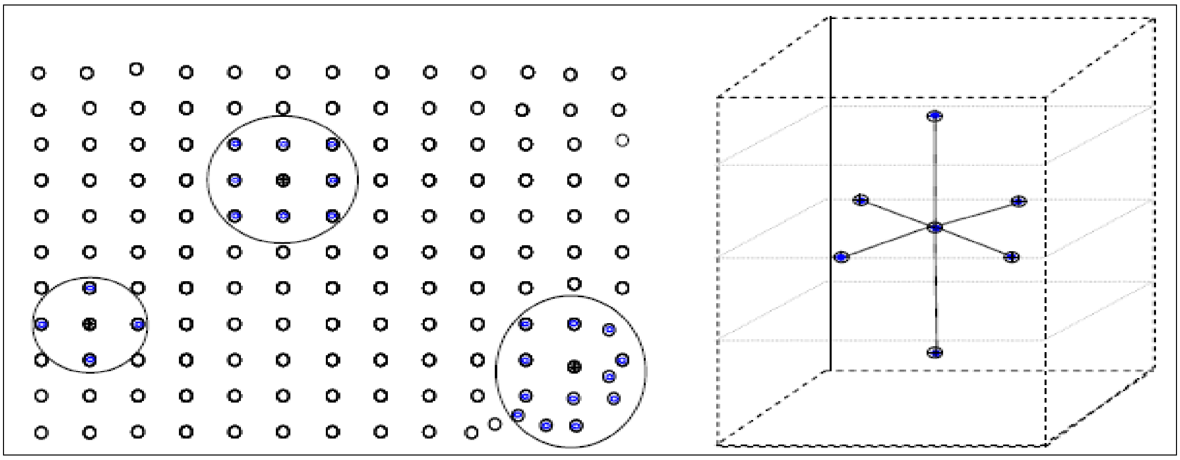

Fig 2: Schematic showing five-nodes, nine-nodes and thirteen-nodes local domains Ω_T^s for 2D (left) and seven nodes local domain for 3D (right)

Using collocation method, Eq. (18) is then applied to all collocation points $\{\hat{x}_k^{[s]}\}_{k=1}^{n_s}$ belonging to the influence domain Ω_T^s of $\hat{x}_s$, we have the following $n_s \times n_s$ linear system

$$\hat{\mathbf{V}}^{[s]} = \mathbf{\Phi}^{[s]}\boldsymbol{\alpha}^{[s]}, \tag{19}$$

Where

$$\mathbf{\Phi}^{[s]} = \left[\Phi\left(\|\hat{x}_i^{[s]} - \hat{x}_j^{[s]}\|\right)\right]_{1\le i,j\le n_s}, \quad \boldsymbol{\alpha}^{[s]} = \left[\alpha_1^{[s]}, \alpha_2^{[s]}, \dots, \alpha_{n_s}^{[s]}\right] \text{ and } \hat{\mathbf{V}}^{[s]} = \left[V\left(\hat{x}_1^{[s]}\right), V\left(\hat{x}_2^{[s]}\right), \dots, V\left(\hat{x}_{n_s}^{[s]}\right)\right].$$

Then, the problem of seeking the expansion coefficients $\{\alpha_k\}_{k=1}^{n_s}$ is transformed into a determination of the values of solution $\hat{V}^{[s]}$ at each center point $\{\hat{x}_k^{[s]}\}_{k=1}^{n_s} \subset \Omega_T^s$ by using the equation

$$\boldsymbol{\alpha}^{[s]} = \left(\mathbf{\Phi}^{[s]}\right)^{-1}\hat{\mathbf{V}}^{[s]}. \tag{20}$$

Hence, Eq. (18) can be written as:

$$\begin{aligned}\hat{\mathbf{V}}^{[s]}(\hat{x}_s) &= (\Phi(\|\hat{x}_s - \hat{x}_1^{[s]}, \Phi(\|\hat{x}_s - \hat{x}_2^{[s]}, \dots, \Phi\left(\|\hat{x}_s - \hat{x}_{n_s}^{[s]}\right)\left(\mathbf{\Phi}^{[s]}\right)^{-1}\hat{\mathbf{V}}^{[s]} \\ &= \Psi^{[s]}\hat{\mathbf{V}}^{[s]}\end{aligned} \tag{21}$$

Where,

$$\Psi^{[s]} = \left(\Phi\left(\|\hat{x}_s - \hat{x}_1^{[s]}\right), \Phi\left(\|\hat{x}_s - \hat{x}_2^{[s]}\right), \dots, \Phi\left(\|\hat{x}_s - \hat{x}_{n_s}^{[s]}\right)\right)\left(\mathbf{\Phi}^{[s]}\right)^{-1}$$

By padding the vector $\Psi^{[s]}$ with zeros based on the mapping between $\hat{\mathbf{V}}^{[s]}$ and $\hat{V}$ we get a new vector Ψ with N components. We then formulate Eq. (21) in terms of global $\hat{\mathbf{V}}$ as

$$\hat{\mathbf{V}}(\hat{x}_s) = \Psi\hat{\mathbf{V}} \tag{22}$$

The local approximation of equations in the system (17) can be determined by applying the differential operators $\mathbf{L}_{\hat{x}}$ and $\mathbf{B}_{\hat{x}}$ to the equation (18) for any selected center point $\hat{x}_s$ in any sub-domain Ω_s. For $\hat{x}_s \in \Omega_s$, we obtain the following equation

$$\begin{aligned}\mathbf{L}_{\hat{x}}\hat{V}(\hat{x}_s) &= \textstyle\sum_{k=1}^{n_s}\alpha_k \mathbf{L}_{\hat{x}}\Phi\left(\|\hat{x}_s - \hat{x}_k^{[s]}\|\right) \\ &= \mathbf{\Upsilon}^{[s]}\,\hat{\mathbf{V}}^{[s]}.\end{aligned} \tag{23}$$

In the same way, for a center $\hat{x}_s$ on the boundary $\partial\Omega_T$, we have

$$\begin{aligned}\mathbf{B}_{\hat{x}}V(\hat{x}_s) &= \textstyle\sum_{k=1}^{n_s}\alpha_k \mathbf{B}\Phi\left(\|\hat{x}_s - \hat{x}_k^{[s]}\|\right) \\ &= \mathbf{\Theta}^{[s]}\,\hat{\mathbf{V}}^{[s]},\end{aligned} \tag{24}$$

Where $\hat{\mathbf{V}}^{[s]} = \left[V(\hat{x}_1^s), V(\hat{x}_2^s), \cdots, V(\hat{x}_{n_s}^s)\right]$, $\mathbf{\Upsilon}^{[s]} = \mathbf{L}_{\hat{x}}\mathbf{\Phi}^{[s]}\left(\mathbf{\Phi}^{[s]}\right)^{-1}$ and $\mathbf{\Theta}^s = \mathbf{B}\mathbf{\Phi}^{[s]}(\mathbf{\Phi}^{[s]})^{-1}$

To switch from local systems (23) and (24) to global ones, the vector $\hat{\mathbf{V}} = [V(\hat{x}_1), V(\hat{x}_2), \cdots, V(\hat{x}_N)]$ is incorporated in the systems (23-24) by adding zeros at the proper locations based on the mapping of $\hat{\mathbf{V}}^{[s]}$ to $\hat{\mathbf{V}}$, and considering $\mathbf{\Upsilon}_{1\times N}$ and $\mathbf{\Theta}_{1\times N}$ as the global expansions of $\mathbf{\Upsilon}_{1\times n_s}^{[s]}$ and $\mathbf{\Theta}^{[s]}{}_{1\times n_s}$ respectively.
The global system of Eq. (23) and (24) are then written under the form:

$$f(\hat{x}_s) = \mathbf{L}_{\hat{x}}\hat{V}(\hat{x}_s) = \mathbf{\Upsilon}(\hat{x}_s)\hat{\mathbf{V}} \tag{25}$$

and

$$h(\hat{x}_s) = \mathbf{B}\hat{V}(\hat{x}_s) = \mathbf{\Theta}(\hat{x}_s)\hat{\mathbf{V}}. \tag{26}$$

By collocating at all domain centers points $\{\hat{x}_j\}_{j=1}^{N_{in}}$, where N_{in} is their number, using Eq. (25) and collocating at all boundary points $\{\hat{x}_j\}_{j=N_{in}+1}^{N}$ using (26), we get the following final sparse linear system of equations

$$\begin{bmatrix} \Upsilon \\ \Theta \end{bmatrix} \begin{bmatrix} \hat{V} \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix}$$

Where:

$$\Upsilon = \begin{bmatrix} \Upsilon(\hat{x}_1) \\ \vdots \\ \Upsilon(\hat{x}_{N_{In}}) \end{bmatrix} \text{ and } \Theta = \begin{bmatrix} \Theta(\hat{x}_{N_{In}+1}) \\ \vdots \\ \Theta(\hat{x}_N) \end{bmatrix}$$

The approximate solution at the interpolation points $\{\hat{V}(\hat{x}_j)\}_{j=1}^{N}$ can be obtained by solving the above sparse linear system of equations.

4. Numerical Experiments

In this section, we investigate the numerical solution of the European and American put options using space-time localized RBF collocation method to show its efficiency and accuracy for solving such problem. The Black-Scholes equation can be solved into two different ways. It can be solved either directly by considering it as a partial differential equation with variable coefficients or as partial differential equation with constant coefficients. Many different examples are solved herein and the results obtained are compared to some numerical published results. The accuracy of the numerical tests is measured using the root mean squared error (RMSE) defined by

$$RMSE = \sqrt{\frac{1}{N}\sum_{j=1}^{N}(\hat{V}_j - V_j)^2},$$

Where V_j and $\hat{V}_j$ are the exact and the approximate solution of the option pricing problem treated at the N collocation points (x_j, t_j) respectively and d is the space dimension of the problem domain.

4.1 Numerical Computation of European Options

To validate the proposed method face to option pricing problems, we first solve the European option problem represented by two examples 1 and 2.

Example 1

For this first example, results obtained are compared to the exact solution and also to some published results that using meshfree method combined with explicit time integration scheme as the explicit first order backward difference (BD1), second order Runge-Kutta (RK2) and fourth order Runge-Kutta (RK4) methods [8]. All these methods are based on the use of radial basis functions as a basis functions for the collocation scheme and solved the problem in the domain $\Omega = [1,30]$ at each time level. Their chosen parameters are $E = 10$, $\sigma = 0.2$, $r = 0.05$ and the number of used centers in the x-axe is $N_x = 81$. For comparison we use the same problem's parameters and we set the space-time domain and centers to be $\Omega_T = [1,30] \times [0,0.5]$ and $N = N_x \times N_t = 81 \times 30$ respectively. Where N_t is the number of points in the time axe.

The problem is solved under two different forms. On one hand as a PDE with variable coefficients and on the other hand as a PDE with constant coefficients. For numerical test, the MQ-RBF is taken under the form $\sqrt{(\epsilon r)^2 + 1}$ and the shape parameter of the MQ-RBF is chosen to be $\epsilon = 0.056$ for PDEs with variable coefficients and $\epsilon = 0.013$ for PDEs with constant coefficients case. The number of nearest nodes is set to be $n_s = 5$ in the entire space-time domain. The comparison of these numerical solutions with the exact, second order and fourth order Runge-Kutta method and the explicit first order backward difference (BD1) solutions is shown in Table 1.

Table 1: Comparison of results obtained using space-time localized RBF collocation method to exact and other numerical schemes [8] for the European option, Example 1

Stock S	Exact	Explicit Time Schemes [8]			Space-Time Scheme	
		RK4	RK2	BD1	Variable coefficients	Constant coefficients
					ϵ=0.056	ϵ= 0.013
2	7.7531	7.7531	7.7531	7.7531	7.7545	7.7532
4	5.7531	5.7531	5.7531	5.7531	5.7531	5.7533
6	3.7532	3.7532	3.7531	3.7533	3.7532	3.7534
7	2.7568	2.7568	2.7564	2.7566	2.7572	2.7574
8	1.7987	1.7984	1.7979	1.7985	1.7999	1.7994
9	0.9880	0.9873	0.9881	0.9902	0.9872	0.9860
10	0.4420	0.4412	0.4429	0.4457	0.4400	0.4383
11	0.1606	0.1602	0.1612	0.1627	0.1599	0.1581
12	0.0483	0.0482	0.0482	0.0485	0.0482	0.0475
13	0.0124	0.0123	0.0121	0.0119	0.0127	0.0124
14	0.0028	0.0028	0.0026	0.0024	0.0030	0.0029

15	0.0006	0.0006	0.0005	0.0004	0.0007	0.0006
16	0.0001	0.0001	0.0001	0.0001	0.0002	0.0001
RMSE		3.28E-04	4.06E-04	1.35E-03	8.25E-04	1.40E-03

According to these results, the proposed method is comparable to others. We can mention that the benefit is that we solve the problem only once in our case. Finally, we can conclude that good accuracy are obtained for the two formulations, variable and constant coefficients, when solving Black-Scholes equation in the case of European put options using space-time localized RBF method.

Example 2

One of the advantage of the proposed technique over other time integration schemes is that the stability analysis for time discretization is no longer needed. To show that, we compute the numerical solutions of the European put option by using different values of N_x and N_t such that the stability condition of explicit time integration schemes, for some given N_x and N_t, cannot provide a convergent solution. The quotient $\frac{\Delta x}{\Delta t}$ has many different values great and less than one ($\frac{\Delta x}{\Delta t} < 1$ and $\frac{\Delta x}{\Delta t} > 1$). Two simulations are given for this example. In the first simulation we set $N_t = 60$ and we take different values of N_x. And in the second simulation we set $N_x = 81$ and we choose different values of N_t. The shape parameter value of the MQ radial function used in the two simulation is set to be $\epsilon = 1.2$ and $\epsilon = 1$ respectively. The space-time domain is $\Omega_T = [1,30] \times [0,0.5]$. For this example 2 the treated PDE is considered under the form of constant coefficients.

Results obtained and their comparison to the exact solution are depicted in Tables 2 and 3. We can then observe from these given results that the numerical accuracy of the space-time scheme is comparable to the analytical solution. So, it can be remarked that the stability of the scheme is asserted and the error RMSE is decreasing with respect to the N_x and N_t values. We can than concluded that the space-time method shows a reasonably good approximation to the solution for these different values of N_x and N_t.

Table 2: Comparison of results obtained by the space-time schemes for Example 2 - Simulation 1, with different N_x and $N_t = 60$ using the constant coefficients formulation

Stock S	Exact	Space-Time scheme				
		$N_x = 41$	$N_x = 61$	$N_x = 81$	$N_x = 101$	$N_x = 121$
2	7.7531	7.7513	7.7523	7.7526	7.7527	7.7528
4	5.7531	5.7509	5.7521	5.7528	5.7528	5.7529
6	3.7532	3.7515	3.7529	3.7528	3.7530	3.753
7	2.7568	2.7572	2.7574	2.7570	2.7570	2.7570
8	1.7987	1.8015	1.8013	1.7994	1.7998	1.7993
9	0.9880	0.9780	0.9899	0.9867	0.9882	0.9878
10	0.4420	0.4246	0.4423	0.4393	0.4414	0.4411
11	0.1606	0.1493	0.1603	0.1588	0.1606	0.1603
12	0.0483	0.0452	0.0503	0.0480	0.0486	0.0486
13	0.0124	0.0128	0.0133	0.0127	0.0126	0.0125
14	0.0028	0.0037	0.0036	0.0031	0.0030	0.0029
15	0.0006	0.0011	0.0009	0.0007	0.0006	0.0006
16	0.0001	0.0003	0.0002	0.0001	0.0001	0.0001
RMSE		6.57E-03	1.18E-03	1.02E-03	4.03E-04	3.53E-04

Table 3: Comparison of results obtained by the space-time schemes for Example 2 - Simulation 2, with different N_t and $N_x = 81$ using the constant coefficients formulation

Stock S	Exact	Space-Time scheme				
		$N_t = 50$	$N_t = 70$	$N_t = 90$	$N_t = 110$	$N_t = 130$
2	7.7531	7.7528	7.7528	7.7528	7.7529	7.7528
4	5.7531	5.7529	5.7529	5.7529	5.7529	5.7529
6	3.7532	3.7530	3.7529	3.7529	3.7529	3.7529
7	2.7568	2.7572	2.7572	2.7572	2.7572	2.7572
8	1.7987	1.7995	1.7995	1.7995	1.7995	1.7995
9	0.9880	0.9865	0.9867	0.9867	0.9868	0.9868
10	0.4420	0.4390	0.4393	0.4393	0.4394	0.4394
11	0.1606	0.1585	0.1587	0.1588	0.1588	0.1588
12	0.0483	0.0478	0.0480	0.0480	0.0480	0.0480
13	0.0124	0.0126	0.0127	0.0127	0.0127	0.0127
14	0.0028	0.0030	0.0031	0.0031	0.0031	0.0031
15	0.0006	0.0007	0.0007	0.0007	0.0007	0.0007
16	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
RMSE		1.14E-03	1.03E-03	1.02E-03	9.88E-04	9.90E-04

4.2 Numerical Computation of American Options

The investigation of the method to solve the American put options problem is firstly based on transforming the problem (13) by using the penalty approach into a problem under the form

$$LV(S,t) - p \times \left(h(S) - V(S,t)\right)^{+} = 0, \qquad \forall (S,t) \in \Omega_T,$$

that can be solved iteratively until we get needed tolerance as shown in the algorithm 1 below:

Algorithm 1 New penalty iterative scheme for solving American Options

```
Input p, tol
Initialize V, V0 // Possible guesses: V0=0 and V=h
V0 ← V
for i=1 to Ni
      //Ni is the number of interior nodes
      //Constructing the i-th line of the linear system
      //Using the flowing lines for each case (Eq 27)
      if Vi < hi then
            LVi + p _ Vi = p _ hi
      Else
            LVi = 0
      end if
end for
Constructing the lines of the boundary conditions (BV)
Solve the linear system
```

Example 3

Most of the chosen parameters of this first considered American put option example are extracted from reference [9]. They are $E = 100$, $r = 0.1$, $\sigma = 0.30$ and $T = 1$. The truncated domain is taken to be $\Omega = [1{,}450]$ for variable coefficients problem, so that $[0, \ln(450)]$ is the domain for the constant coefficients PDE. During the experiments tests we take $N_x = 101$, $N_t = 100$ and the penalty parameter is set to be $p = 1000$. Two different shape parameters are chosen $\epsilon = 1.2$ and $\epsilon = 1.41$. The convergence criteria of the penalty algorithm 1 is taken to be $tol = 1e-3$. Following [9] the artificial boundary condition are used for the treatment of the considered example 3. The problem is solved iteratively following the algorithm 1 until the accuracy needed is achieved. In Table 4, we show the results obtained and the results of Binomial method and those published in [9] using RBFs method with explicit scheme. Taken the results obtained using Binomial method as a reference, it can be concluded from Table 4 that although the discretization time step of the explicit scheme adopted in is so small, since $N_t = 1000$, the RMSE error of the space-time scheme is less than that obtained with explicit scheme. So, we can conclude that the space-time scheme gives a good approximation to the American put option compared to results given in [9]. At $S = E$, Table 4 shows that the accurate result compared to the Binomial method, as a reference, is achieved with the space-time scheme. For more comparison we investigated another numerical test with small values of $N_x = 30$ and $N_t = 81$ with a truncated space domain $\Omega = [1{,}300]$ which is small that the first taken one. The RMSE obtained for this last simulation of example 3, is 0.0137 which is better than results found in [9] for the simulation with $N_x = 101$ and $N_t = 100$ and almost the same as the RMSE obtained with $N_x = 101$ and $N_t = 1000$. For our last test the shape parameter of the MQ-RBF is chosen $\epsilon = 0.005$. All simulations are done using constant coefficients formulation of the PDE.

Table 4: Comparison of accuracy for the American option with $E = 100$, $r = 0.1$, $\sigma = 0.30$, $T = 1$, $\Omega = [1{,}450]$ $N_x = 101$, $N_t = 100$.

Stock S	Binomial	Space-Time Scheme		RBFs method with explicit schemes	
		$\epsilon = 1.2$	$\epsilon = 1.41$	$N_t = 100$	$N_t = 1000$
80	20.2689	20.2613	20.2650	20.2916	20.2889
85	16.3467	16.3344	16.3407	16.3715	16.3644
90	13.1228	13.1095	13.1176	13.1446	13.1346
95	10.4847	10.4747	10.4835	10.5077	10.4956
100	8.3348	8.3246	8.3316	8.3608	8.3482
105	6.6071	6.5961	6.6027	6.6257	6.6133
110	5.2091	5.2090	5.2147	5.2299	5.2186
115	4.0976	4.1014	4.1057	4.1131	4.1034
120	3.2059	3.2105	3.2122	3.2246	3.2167
140	1.1789	1.1925	1.1879	1.1860	1.185
160	0.4231	0.4320	0.4251	0.4237	0.426
180	0.1502	0.1545	0.1496	0.1491	0.1518
200	0.0529	0.0537	0.0522	0.0522	0.0543
RMSE		0.0089	0.0051	0.0180	0.0107

Example 4 Two-Asset American Put

For this example, we apply the penalty method developed in the previous section to compute the two-asset American put option price. The problem has the form

$$\frac{\partial V}{\partial t}-\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 V}{\partial S_1^2}-\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 V}{\partial S_2^2}-(r-D_1)\frac{\partial V}{\partial S_1}-(r-D_2)\frac{\partial V}{\partial S_2}+rV+p\times(h-V)^+=0,$$

with $V(S_1,S_2,0)=h(S_1,S_2)=\max(E-\alpha_1 S_1-\alpha_2 S_2,0)$ as initial boundary condition. The used parameter values for this considered problem are $r=0.1$, $T=1$; $E=1$, $D1=0.05$, $\sigma_1=0.2$, $\alpha_1=0.6$, $D2=0.01$, $\sigma_2=0.3$, $\alpha_2=1-\alpha_1$ and the truncated space domain Ω is $[0,1]^2$. For other numerical parameters values, we chose $N_x=N_y=N_t=20$, the shape parameter $\epsilon=0.1$ and the number of nearest points is chosen to be $n_s=7$. The penalty parameter is taken to be $p=1000$ and the tolerance is $tol=1e-5$.

In many works artificial boundary conditions like the linear boundary condition or the far-field and near-field boundary conditions were used. A technique that is common in financial industry. Another choice of the boundary conditions is $V|_{S_i=\infty}=0$ and $V|_{S_i=0}=g_i$, for $i=1,\dots,d$, where g_i, $\forall i=1,\dots,d$ are determined by solving the associated $(d-1)$-asset of American put option pricing problem. This last choice is not adopted herein because of the fact that there is a possibility that the errors in lower dimensions can be transformed and enlarged in higher dimensions, and also these choices are artificial. So, In herein we follow [12, 14, 20] and solve the problem without setting any conditions on the space boundary $\partial\Omega$. Janson and Tysk [20] have shown that the problem we consider here is actually well posed without boundary conditions as long as the growth at infinity is restricted. To numerically validate our choice of no boundary conditions, we compare the single-asset American put option to the two-asset option price at the boundary $S_2=0$.The PDE is solved under its original form as PDE with variable coefficients without making any variable change. Fig 3 shows the plot of the numerical solution computed at time $\tau=0$ $(t=T)$ and the payoff function. Results of comparison are described by Fig 4. We remark that our results are accurate compared to other published works.

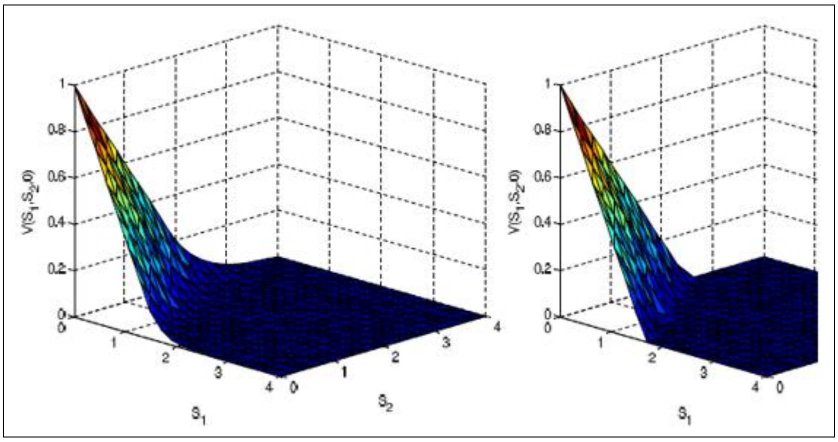

Fig 3: Option price at t = T for two-assets American put option (left) and the payoff function (right)

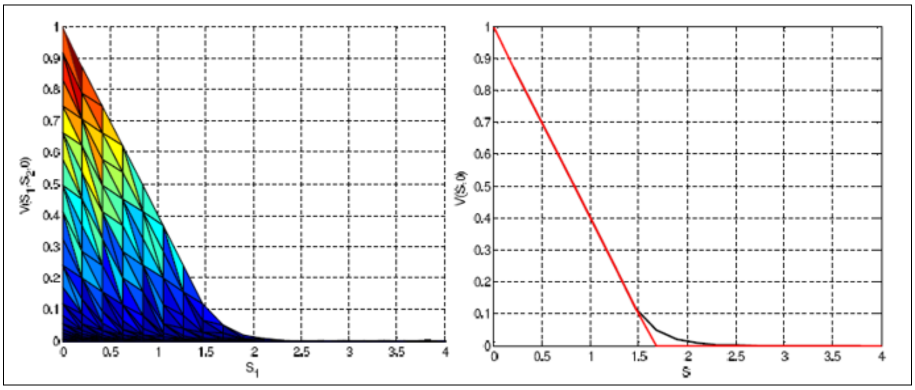

Fig 4: Option price at S2 = 0 and t = T for two-assets American put option (left) and single-asset (right)

Table 5: Comparison of two-asset solution at each boundary $\{S_i=0, i=1,2\}$ to the associated one-asset problem at $t=T$. (MAE=Maximum Absolute Error).

$V-g_2$		$V-g_1$	
MAE	RMSE	MAE	RMSE
1.63E-03	4.68E-04	3.70E-03	1.32E-03

In Table 5, we compare the obtained numerical solution at $S = S_i, i = 1,2$ with those obtained by solving the associated 1D problems. For the condition $V|_{S_i=S_{i_{max}}}$, $i = 1,2$, the numerical value is zero. We can conclude that the choice is valid comparing to these conditions. We can also mention that although the shape parameter is chosen in the large interval $[0.01,10]$ we still have a stable results.

Example 5-Three-Asset American Put

As a last example we apply the proposed technique to compute the three-asset American put option price. The problem considered has the form:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} - \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} - \frac{1}{2}\sigma_3^2 S_3^2 \frac{\partial^2 V}{\partial S_3^2} - (r - D_1)\frac{\partial V}{\partial S_1} - (r - D_2)\frac{\partial V}{\partial S_2} - (r - D_3)\frac{\partial V}{\partial S_3} + rV + p \times (h - V)^+ = 0,$$

With the following initial condition:

$$V(S_1, S_2, S_3, 0) = h(S_1, S_2, S_3) = \max(E - \alpha_1 S_1 - \alpha_2 S_2 - \alpha_3 S_3, 0).$$

The parameter values of this considered problem are given as
$r = 0.1,\ T = 1,\ E = 1, D1 = 0.05, \sigma_1 = 0.2, \alpha_1 = 0.3, D2 = 0.01, \sigma_2 = 0.3, \alpha_2 = 0.3, D3 = 0.01, \sigma_3 = 0.2$ and $\alpha_3 = 0.4$. For this example, the truncated space domain is $\Omega = [0,4]^3$ and $N_x = N_y = N_z = N_t = 11$, which means that the number of collocation points in space-time domain Ω_T is $N = 11^4$. The number of nearest nodes and the shape parameter are $n_s = 9$ and $\epsilon = 0.04$ respectively. The detail about the choice of the number of nearest point for a problem with d -dimension is given in. The penalty parameter is set to be $p = 1000$ and the convergence criteria of the penality algorithm 1 is $tol = 1e - 5$. The problem is solved without any space boundary condition as it has been done for the two-asset American put option (Example 4). Again, to validate numerically our choice of no boundary conditions, we compare the three-asset solution at each boundary to the associated two-asset problem using the same parameters. Table 6 depicted the results of comparison. In Table 7, we compare the numerical solution to the condition $V = 0$ at each boundary with $S_i = S_{i_{max}}$, for $i = 1,2,3$. Table 7, Table 6 and Fig. [Fig 5] show that we have acceptable results and proving the validity of our choice.

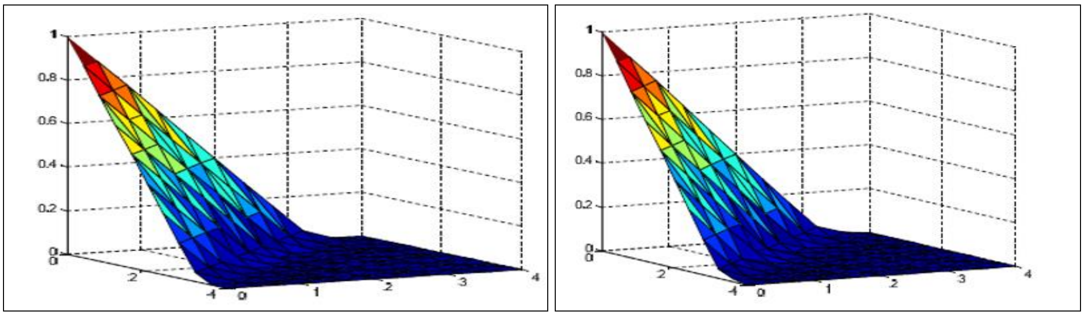

Fig 5: Two-asset numerical solution (left); Three-asset numerical solution with $S_3 = 0$ (right)

Table 6: Comparison of three-asset solution at each boundary $\{S_i = 0, i = 1,2,3\}$ to the associated two-asset problems at $t = T$

$V - g_3$		$V - g_2$		$V - g_1$	
MAE	RMSE	MAE	RMSE	MAE	RMSE
1.24E-02	2.60E-03	1.90E-02	3.18E-03	2.19E-02	2.58E-03

Table 7: Comparison of three-asset solution at each boundary to the condition $V = 0$ at $t = T$

$V_{S_1=S_{1_{max}}}$		$V_{S_2=S_{2_{max}}}$		$V_{S_3=S_{3_{max}}}$	
$\Vert V \Vert_\infty$	$\Vert V \Vert_2$	$\Vert V \Vert_\infty$	$\Vert V \Vert_2$	$\Vert V \Vert_\infty$	$\Vert V \Vert_2$
3.21E-05	4.66E-06	4.66E-05	7.36E-06	2.21E-04	2.29E-05

5. Conclusion

Application of the space-time localize RBFs method, developed in, to the options pricing is presented. This technique is based on the local radial basis function without differentiating between space and time variables. Unlike the other methods, the proposed method don't need any time discretization and solve the problem as a new formulated one. The system matrix obtained is sparse which ameliorate the speed and the accuracy when solving the algebraic system. It also benefit from the following advantages:

1. Time discretization as implicit, explicit, θ-method, method-of-line approach and other are not applied,
2. The time stability analysis is not discussed,
3. Re-computation of the resulting matrix at each time level as done for other method for solving partial differential equations (PDEs) with variable coefficients is avoided and the matrix is computed once.

The problem of American options is treated as a linear complementary problem and solved as a penalty one in an iterative way. The results of two- and three-asset American option price presented in this paper indicate that such artificial boundary conditions are not, in fact, necessary. Numerical results show that this space-time localized radial basis functions method, offers good accuracy in the computation of both the European and the American Options. The good results obtained with this new numerical scheme prove that this technique presents an alternative computational algorithm to solve higher dimension time-depend financial problem.

6. References

1. Hamaidi M, Naji A, Charafi A. Space-time localized radial basis function collocation method for solving parabolic and hyperbolic equations, Engineering Analysis with Boundary Elements. 2016; 67:152-163.
2. Hamaidi M, Naji A, Taik A. Solving Parabolic and Hyperbolic Equations with Variable Coefficients Using Space-Time Localized Radial Basis Function Collocation Method, Modelling and Simulation in Engineering, 2021. Article ID 6688806.
3. Hamaidi M, Naji A, Ghafrani F, Jourhmane M. Noniterative Localized and Space-Time Localized RBF Meshless Method to Solve the Ill-Posed and Inverse Problem, Modelling and Simulation in Engineering, 2020. Article ID 5046286.
4. Fischer Black, Myron Scholes. The Pricing of Options and Corporate Liabilities, Journal of Political Economy. 1974; 81(3):637-654.
5. Ran Zhang, Qi Zhang, Haiming Song. An efficient finite element method for pricing American multi-asset put options, Communications in Nonlinear Science and Numerical Simulation. 2015; 29:25-36.
6. Wang S, Yang XQ, Teo KL. Power Penalty Method for a Linear Complementarity Problem Arising from American Option Valuation. Journal of Optimization Theory and Applications. 2006; 129(2):227-254.
7. Alexane Verraux. Résolution de l'équation de Black et Scholes-Thèse scientifique, Université de Technologie de Tampere, 2008.
8. Hon YC, Mao XZ. A radial basis function method for solving options pricing model. Journal of Financial Engineering. 1999; 8(1):1-24.
9. Zongmin WU, Hon YC. Convergence error estimate in solving free boundary diffusion problem by radial basis functions method; Engineering Analysis with Boundary Elements. 2003; 27:73-79.
10. Hon YC. A quasi radial basis functions method for American option pricing. Computer Math Appl. 2002; 43(3):513-524.
11. Fasshauer GE, Khaliq AQM, Voss DA. Using meshfree approximation for multi-asset American options. J Chin Inst Eng. 2004; 27:563-571.
12. Marcozzi MD, Choi S, Chen CS. On the use of boundary conditions for the variational formulations arising in financial mathematics. Appl Math Comput. 2001; 124:197-214.
13. Golbabai Ahamad, Ahmadian Davood, Milev Mariyan. Radial basis function with application to finance: American put option under jump diffusion. Math Comput Model. 2012; 55:1354-1362.
14. Ulrika Pettersson, Elisabeth Larsson, Gunnar Marcusson, Jonas Persson. Improved radial basis function methods for multi-dimensional option pricing, Journal of Computational and Applied Mathematics. 2008; 222:82-93.
15. Jamal Amani Rad, Kourosh Parand, Saeid Abbasbandy. Local weak form meshless techniques based on the radial point interpolation (RPI) method and local boundary integral equation (LBIE) method to evaluate European and American options, Communications in Nonlinear Science and Numerical Simulation. 2015; 22:1178-1200.
16. Jamal Amani Rad, Kourosh Parand, Luca Vincenzo Ballestra. Pricing European and American options by radial basis point interpolation, Applied Mathematics and Computation. 2015; 251:363-377.
17. Joonglee Jo, Yongsik Kim. Comparison of numerical schemes on multi-dimensional Black-Scholes equations, Korean Mathematical Society, 2013, 2035-2051.
18. Larsson Elisabeth, Ahlander Krister, Hall Andreas. Multi-dimensional option pricing radial basis functions and the generalized Fourier transform. Journal Comput Appl Math. 2008; 222:175-192.
19. Martin-Vaquero J, Khaliq AQM, Kleefeld B. Stabilized explicit Runge-Kutta methods for multi-asset American options, Computers and Mathematics with Applications. 2014; 67:1293-1308.
20. Janson S, Tysk J, Feynman Kac. Formulas for Black-Scholes type operators, Bull. London Math. Soc. 2006; 38:269-282.
21. Li Z, Mao XZ. Global space-time multiquadric method for inverse heat conduction problems, International Journal for Numerical Methods in Engineering. 2011; 85(3):355-379.
22. Li Z, Mao XZ. Global multiquadric collocation method for groundwater contaminant source identification, Environmental Modelling & Software. 2011; 26:1611-1621.
23. Li TS, Wong SM. Development of an Efficient and Accurate Global Space-Time Radial Basis Collocation Model for Estimation of River Pollution Source, IACSIT International Journal of Engineering and Technology. 2014; 6(2).
24. Myers DE, De Iaco S, Posa D, De Cesare L. Space-time radial basis functions, Computers and Mathematics with Applications. 2002; 43:539-549.
25. Abdul QM, Khaliq, David A, Voss, Kamran Kazmi. Adaptive θ-methods for pricing American options, Journal of Computational and Applied Mathematics. 2008; 222:210-227.
26. Bjørn Fredrik Nielsen, Ola Skavhaug, Aslak Tveito. Penalty methods for the numerical solution of American multi-asset option problems, Journal of Computational and Applied Mathematics. 2008; 222:3-16.
27. Forsyth PA, Vetzal K. Quadratic convergence for valuing American options using a penalty method, SIAM J. Sci. Comput. 2002; 23(6):2095-2122.