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On *Pseudo T* - Birecurrent Finsler Space in Berwald Sense

[1] Abdalstar A Saleem, [2] Alaa A Abdallah
[1] Department of Mathematics, Faculty of Sciences, Aden University, Aden, Yemen
[2] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India

Corresponding Author: **Alaa A Abdallah**

Abstract
In this present paper, we introduce a Finsler space which Pseudo curvature tensor satisfies the birecurrence property in sense of Berwald. Also, we prove that a flat pseudo $T-$birecurrent space for H^i_{jkh} exist. Certain identities belong to this space have been studied. Finally, the projection on indicatrix with respect to Berwald's connection for the tensors whose behave as birecurrent have been discussed.

Keywords: Pseudo $T-$Birecurrent Space, Flat Pseudo $T-$Birecurrent Space, Projection on Indicatrix

1. Introduction

The recurrence property has been studied by the Finslerian geometrics. Sinha [12] introduced the torsion tensor T^i_{jk} and curvature tensor T^i_{jkh} from the deviation tensor T^i_j. Dabey and Singh [2] and Pandey and Dwivedi [4, 5] considered the space equipped curvature tensor T^i_{jkh} is recurrent and called it $T-$recurrent Finsler space. They also considered there in projectively flat $-T-$recurrent space and obtained certain results belong to these spaces. Saleem [9] studied the flat of recurrent curvature tensor fields in Finsler space. Further, Qasem and Saleem [7], Pandey and Verma [6], Saleem and Abdallah [10], Singh [11] and Sinha [12] were studied on birecurrent curvature tensor fields in Finsler space. Saleem and Abdallah [10] study the projection on indicatrix for some tensors whose satisfy the birecurrence property.

Let us consider an $n-$dimensional Finsler space F_n equipped with the line elements (x, y) and the fundamental metric function F is positively homogeneous of degree one in y^i. Berwald covariant derivative $\beta_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [1, 8]

$$\beta_k T^i_j = \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} .$$

The processes of Berwald covariant differentiation with respect to x^h and the partial differentiation with respect to y^k commute according to

(1.1) a) $\dot{\partial}_j \beta_l T^i_h - \beta_l \dot{\partial}_j T^i_h = T^r_h G^i_{klr} - T^i_r G^r_{klh}$,

where the tensor G^i_{jkh} is symmetric in their lower indices and defined by

(1.1) b) $G^i_{jkh} y^j = 0$.

Berwald's covariant derivative of the vectors y^j and l^j vanish identically, i. e.

(1.2) a) $\beta_l y^j = 0$ and b) $\beta_l l^j = 0$,

where

c) $l^j = \dfrac{y^j}{F}$.

Definition 1.1: The projection of any tensor T_j^i on indicatrix is given by [3, 10]

(1.3) $p.T_j^i = T_\beta^\alpha h_\alpha^i h_j^\beta$,

Where the angular metric tensor is homogeneous function of degree zero in y^i and defined by

(1.4) $h_j^i = \delta_j^i - l^i l_j$.

Definition 1.2: If the projection of a tensor T_j^i on indicatrix I_{n-1} is the same tensor T_j^i , then the tensor is called an indicatrix tensor or an indicatory tensor.

2. Preliminaries

In this section, we introduce some important concepts and definitions. The pseudo deviation tensor T_j^i is positively homogeneous of degree 2 in y^i and defined by [12]

$$T_j^i = -\left\{H\delta_j^i + \frac{1}{n+1}\left(\dot\partial_r H_j^r - \dot\partial_j H\right)y^i\right\}.$$

The pseudo torsion tensor T_{jk}^i is positively homogeneous of degree 1 in y^i and defined by

$$T_{jk}^i = \frac{1}{n+1}\left\{y^i H_{rkj}^r + 2\delta_{[j}^i\left(H_{k]} + \dot\partial_{[k} H\right)\right\}.$$

The pseudo curvature tensor T_{jkh}^i is positively homogeneous of degree 0 in y^i and defined by

$$T_{jkh}^i = \frac{1}{n+1}\left\{\delta_{[j}^i H_{rkh}^r + y^i\dot\partial_j H_{rkh}^r + 2\delta_{[k}^i\left(H_{(j)h} + \dot\partial_{h]}\dot\partial_j H\right)\right\}.$$

These tensors satisfy the following [12]

(2.1) a) $T_{jkh}^i y^j = T_{kh}^i$, b) $T_{jkh}^i = \dot\partial_j T_{kh}^i$ c) $T_{kh}^i y^k = T_h^i$ and d) $T_{jkh}^i = W_{jkh}^i - H_{jkh}^i$,

where the curvature tensor H_{jkh}^i , torsion tensor H_{jk}^i and deviation tensor H_j^i are positively homogeneous of degree zero, one and two in y^i , respectively. And satisfy the following

(2.2) a) $H_{jkh}^i y^j = H_{kh}^i$, b) $H_{kh}^i y^k = H_h^i$, c) $H_{jki}^i = H_{jk}$, d) $H_{jki}^i = H_{jk}$, e) $H_k y^k = (n-1)H$ and f) $\dot\partial_j H_k = H_{jk}$.

The Bianchi identity for the curvature tensor H_{jkh}^i is given by

(2.2) g) $H_{jkh}^i + H_{khj}^i + H_{hjk}^i = 0$.

The projective curvature tensor W_{jkh}^i , torsion tensor W_{jk}^i and deviation tensor W_j^i are positively homogeneous of degree zero, one and two in y^i , respectively. And satisfy the following [8]

(2.3) a) $W_{jkh}^i y^j = W_{kh}^i$, b) $W_{kh}^i y^k = W_h^i$ and c) $W_{jki}^i = 0$.

The Bianchi identity for the projective curvature tensor W_{jkh}^i is given by [11]

(2.3) d) $W_{jkh}^i + W_{khj}^i + W_{hjk}^i = 0$.

A Finsler space called a pseudo $T-$recurrent space if the curvature tenser T_{jkh}^i satisfies [4, 5]

(2.4) $\beta_l T_{jkh}^i = \lambda_l T_{jkh}^i$, $T_{jkh}^i \neq 0$,

where λ_l is non-zero covariant vector field. Since the Finsler space is projective flat, then we have [8]

(2.5) a) $W_{jkh}^i = 0$, b) $W_{kh}^i = 0$ and c) $W_j^i = 0$.

3. Pseudo $T-$ birecurrent space

In this section, we introduce a Finsler space which the curvature tensor T_{jkh}^i is birecurrent in sense of Berwald. Also, we obtained flat pseudo $T-$birecurrent space.

Definition 3.1: A Finsler space F_n which the curvature tensor T_{jkh}^i satisfies the condition

(3.1) $\beta_m \beta_l T_{jkh}^i = a_{lm} T_{jkh}^i$, $T_{jkh}^i \neq 0$, where a_{lm} recurrence covariant tensor field of second order, this space called a *pseudo* $T-$birecurrent space. Differentiating (2.4) covariantly with respect to x^m in sense of Berwald, we get

$$\beta_m \beta_l T^i_{jkh} = (\beta_m \lambda_l) T^i_{jkh} + \lambda_l \beta_m T^i_{jkh} .$$

In view of (2.4), the above equation becomes

$$\beta_m \beta_l T^i_{jkh} = (\beta_m \lambda_l) T^i_{jkh} + \lambda_l \lambda_m T^i_{jkh}$$

Above equation can be written as the condition (3.1) where $a_{lm} = (\beta_m \lambda_l) + \lambda_l \lambda_m$. Thus, we conclude

Theorem 3.1: *Every pseudo $T-$recurrent space which the recurrence vector field satisfies $(\beta_m \lambda_l) + \lambda_l \lambda_m \neq 0$ is a pseudo $T-$birecurrent space.*

Transvecting the condition (3.1) by y^j, using (2.1a) and (1.2a), we get

(3.2) $\beta_m \beta_l T^i_{kh} = a_{lm} T^i_{kh}$.

Transvecting (3.2) by y^k, using (2.1c) and (1.2a), we get

(3.3) $\beta_m \beta_l T^i_h = a_{lm} T^i_h$.

Thus, we conclude

Theorem 3.2: *In pseudo $T-$birecurrent space, the torsion tensor T^i_{kh}, deviation tensor T^i_h are birecurrent.*

Let us consider a Finsler space whose the curvature tensor T^i_{jkh} is a projective flat, i.e. satisfies (2.5). Differentiating (2.1d) covariantly with respect to x^l and x^m in sense of Berwald, using (2.5a) and the condition (3.1), we get

(3.4) $\beta_m \beta_l H^i_{jkh} = a_{lm} H^i_{jkh}$.

Transvecting (3.4) by y^j, using (2.2a) and (1.2a), we get

(3.5) $\beta_m \beta_l H^i_{kh} = a_{lm} H^i_{kh}$.

Transvecting (3.5) by y^k, using (2.2b) and (1.2a), we get

(3.6) $\beta_m \beta_l H^i_h = a_{lm} H^i_h$.

Contracting the indices i and h in (3.4) and using (2.2c), we get

(3.7) $\beta_m \beta_l H_{kh} = a_{lm} H_{kh}$.

Transvecting (3.7) by y^k, using (2.2d) and (1.2a), we get

(3.8) $\beta_m \beta_l H_h = a_{lm} H_h$.

Transvecting (3.8) by y^h, using (2.2e) and (1.2a), we get

(3.9) $\beta_m \beta_l H = a_{lm} H$.

From (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9), we conclude that H^i_{jkh}, H^i_{kh}, H^i_h, H_{kh}, H_h and H satisfy the birecurrence property. Since the Finsler space is flat pseudo, i.e. (2.5a) satisfies in (2.1d). Thus, we conclude

Theorem 3.3: *In flat pseudo $T-$birecurrent space, the curvature tensor H^i_{jkh}, torsion tensor H^i_{kh}, deviation tensor H^i_h, Ricci tensor H_{kh}, curvature vector H_h and curvature secular H are birecurrent.*

Let us consider pseudo $T-$birecurrent space, i.e., characterized by the condition (3.1). Differentiating (3.8) partially with respect to y^k, we get

$$\dot{\partial}_k \beta_m \beta_l H_h = (\dot{\partial}_k a_{lm}) H_h + a_{lm} (\dot{\partial}_k H_h) .$$

Using commutation formula exhibited by (1.1a) for $\beta_l H_h$ in above equation and using (2.2f), we get

$$\beta_m \dot{\partial}_k \beta_l H_h - \beta_r H_h G^r_{klm} - \beta_l H_r G^r_{khm} = (\dot{\partial}_k a_{lm}) H_h + a_{lm} H_{kh} .$$

Again, applying commutation formula exhibited by (1.1a) for H_h in above equation and using (3.7), we get

$$-(\beta_m H_r) G^r_{khl} - H_r (\beta_m G^r_{khl}) - (\beta_r H_h) G^r_{kml} - H_h (\beta_r G^r_{kml}) - \beta_r H_h G^r_{klm} - \beta_l H_r G^r_{khm} = (\dot{\partial}_k a_{lm}) H_h$$

Transvecting above equation by y^l, using (1.2a) and (1.1b), we get

$$-y^l \beta_l H_r G^r_{khm} = y^l \left(\dot{\partial}_k a_{lm} \right) H_h .$$

Transvecting above equation by y^h, using (1.2a), (1.1b) and (2.2e), we get

$$y^l \left(\dot{\partial}_k a_{lm} \right) = 0 ,$$

where $H \neq 0$, which can be written

$$a_{km} = \dot{\partial}_k \left(a_{lm} y^l \right)$$

$$a_{km} = \dot{\partial}_k \left[\dot{\partial}_m \left(a_{ls} y^s \right) y^l \right]$$

$$a_{km} = \dot{\partial}_k \left[\dot{\partial}_m \left(a_{ls} y^s y^l \right) - a_{ms} y^s \right]$$

$$a_{km} = \dot{\partial}_k \dot{\partial}_m \left(a_{ls} y^s y^l \right) - (\dot{\partial}_k a_{sm}) y^s - a_{mk} .$$

Which may be rewritten as

(3.10) $a_{km} + a_{mk} = \dot{\partial}_k \dot{\partial}_m \varnothing ,$

where $\varnothing = a_{ls} y^s y^l$.

Thus, we conclude

Theorem 3.4: *In pseudo $T-$birecurrent space, from (3.10), the symmetric part if the recurrence tensor is birecurrent derivative of the scaler field.*

Differentiating (3.2) partially with respect to y^j, we get

$$\dot{\partial}_j \beta_m \beta_l T^i_{kh} = \left(\dot{\partial}_j a_{lm} \right) T^i_{kh} + a_{lm} \left(\dot{\partial}_j T^i_{kh} \right).$$

Using commutation formula exhibited by (1.1a) for $\beta_l T^i_{kh}$ in above equation and using (2.1b), we get

$$\beta_m \dot{\partial}_j \beta_l T^i_{kh} - \beta_r T^i_{kh} G^r_{jml} + \beta_l T^r_{kh} G^i_{jmr} - \beta_l T^i_{rh} G^r_{jmk} - \beta_l T^i_{kr} G^r_{jmh} = (\dot{\partial} a_{lm}) T^i_{kh} + a_{lm} T^i_{jkh} .$$

Again, applying commutation formula exhibited by (1.1a) for T^i_{kh} in above equation and using (2.1b) and (3.1), we get

$$\left(\beta_m T^s_{kh} \right) G^i_{jls} + T^s_{kh} \left(\beta_m G^i_{jls} \right) - (\beta_m T^i_{sh}) G^s_{jlk} - T^i_{sh} (\beta_m G^s_{jlk}) - (\beta_m T^i_{ks}) G^s_{jlh} - T^i_{ks} \left(\beta_m G^s_{jlh} \right) - \beta_r T^i_{kh} G^r_{jml}$$
$$+ \beta_l T^r_{kh} G^i_{jmr} - \beta_l T^i_{rh} G^r_{jmk} - \beta_l T^i_{kr} G^r_{jmh} = \left(\dot{\partial}_j a_{lm} \right) T^i_{kh} .$$

Transvecting above equation by y^l, using (1.1b) and (1.2a), we get

$$y^l \left(\beta_l T^r_{kh} \right) G^i_{jmr} - y^l \left(\beta_l T^i_{rh} \right) G^r_{jmk} - y^l \left(\beta_l T^i_{kr} \right) G^r_{jmh} = \left(\dot{\partial}_j a_{lm} \right) y^l T^i_{kh} .$$

Taking skew-symmetric part of above equation with respect to the indices l and m, using (1.1b) and (1.2a), we get

(3.11) $y^l \left(\beta_l T^r_{kh} \right) G^i_{jmr} + y^l \left(\beta_l T^i_{rh} \right) G^r_{jmk} + y^l \left(\beta_l T^i_{kr} \right) G^r_{jmh} = 0 .$

Thus, we conclude

Theorem 3.5: *In pseudo $T-$birecurrent space, the skew-symmetric part of the recurrence tensor is the identity (3.11) holds.*

4. Projection on Indicatrix with Respect to Berwald's Connection

In this section, we studied the projection on indicatrix for the tensors which be birecurrent. Let us consider a Finsler space F_n for the curvature tensor T^i_{jkh} is birecurrent in sense of Berwald, i.e. characterized by (3.1). Now, in view of (1.3), the curvature tensor T^i_{jkh} on indicatrix is given by

(4.1) $p.T^i_{jkh} = T^a_{bcd} h^i_a h^b_j h^c_k h^d_h .$

Taking covariant derivative of (4.1) with respect to x^l and x^m in sense of Berwald and using the fact that $\beta_l h^i_j = 0$, then using the condition (3.1) in the resulting equation, we get

$$\beta_m \beta_l \left(p T^i_{jkh} \right) = a_{lm} T^a_{bcd} h^i_a h^b_j h^c_k h^d_h .$$

Using (4.1) in above equation, we get

(4.2) $\beta_m \beta_l \left(p .T^i_{jkh} \right) = a_{lm} \left(p .T^i_{jkh} \right).$

This shows that $p .T^i_{jkh}$ is birecurrent. Thus, we conclude

Theorem 4.1: *The curvature tensor T^i_{jkh} on indicatrix in pseudo $T-$birecurrent space is birecurrent in sense of Berwald.*

Let the projection of curvature tensor T^i_{jkh} on indicatrix is birecurrent, i.e. characterized by (4.2). Using (1.3) in (4.2), we get

$$\beta_m\beta_l\left(T^a_{bcd}h^i_a h^b_j h^c_k h^d_h\right) = a_{lm}T^a_{bcd}h^i_a h^b_j h^c_k h^d_h \,.$$

Using (1.4) in above equation, we get

$$\beta_m\beta_l(T^i_{jkh} - T^i_{jkd}\ell^d\ell_h - T^i_{jch}\ell^c\ell_k + T^i_{jcd}\ell^c\ell_k\ell^d\ell_h - T^a_{jkh}\ell^i\ell_a + T^a_{jkd}\ell^i\ell_a\ell^d\ell_h + T^a_{jch}\ell^i\ell_a\ell^c\ell_k - T^a_{jcd}\ell^i\ell_a\ell^c\ell_k\ell^d\ell_h)$$
$$= a_{lm}(T^i_{jkh} - T^i_{jkd}\ell^d\ell_h - T^i_{jch}\ell^c\ell_k + T^i_{jcd}\ell^c\ell_k\ell^d\ell_h - T^a_{jkh}\ell^i\ell_a + T^a_{jkd}\ell^i\ell_a\ell^d\ell_h + T^a_{jch}\ell^i\ell_a\ell^c\ell_k - T^a_{jcd}\ell^i\ell_a\ell^c\ell_k\ell^d\ell_h)\,.$$

Using (2.1a), (1.2a) and (1.2c) in above equation, we get

$$\text{(4.3)}\quad \beta_m\beta_l(T^i_{jkh} - \frac{1}{F}T^i_{jk}\ell_h - \frac{1}{F}T^i_{jh}\ell_k - T^a_{jkh}\ell^i\ell_a + \frac{1}{F^2}T^i_j\ell_k\ell_h + \frac{1}{F}T^a_{jk}\ell^i\ell_a\ell_h + \frac{1}{F}T^a_{jh}\ell^i\ell_a\ell_k - \frac{1}{F^2}T^a_j\ell^i\ell_a\ell_k\ell_h)$$
$$= a_{lm}(T^i_{jkh} - \frac{1}{F}T^i_{jk}\ell_h - \frac{1}{F}T^i_{jh}\ell_k - T^a_{jkh}\ell^i\ell_a + \frac{1}{F^2}T^i_j\ell_k\ell_h + \frac{1}{F}T^a_{jk}\ell^i\ell_a\ell_h + \frac{1}{F}T^a_{jh}\ell^i\ell_a\ell_k - \frac{1}{F^2}T^a_j\ell^i\ell_a\ell_k\ell_h)\,.$$

Now, since the torsion tensor T^i_{jk} and deviation tensor T^i_j are birecurrent, i.e. satisfy (3.2) and (3.3), respectively. In view of (3.2), (3.3), (1.2b) and (1.2c), then equation (4.3) can be written as

$$\left(T^i_{jkh} - T^a_{jkh}\ell^i\ell_a\right) = a_{lm}\left(T^i_{jkh} - T^a_{jkh}\ell^i\ell_a\right).$$

From last equation, we conclude

Corollary 4.1: *In pseudo $T-$birecurrent space, the projection of the tensor T^i_{jkh} on indicatrix is birecurrent, if and only if $T^a_{jkh}\ell_a$ is birecurrent.*

We know that, the torsion tensor T^i_{jk} is birecurrent, i.e. characterized by (3.2). In view of (1.3), the projection of the torsion tensor T^i_{jk} on indicatrix is given by

$$\text{(4.4)}\quad p\,.T^i_{jk} = T^a_{bc}h^i_a h^b_j h^c_k\,.$$

Taking covariant derivative of (4.4) with respect to x^l and x^m in sense of Berwald and using the fact that $\beta_l h^i_j = 0$, then using (3.2) in the resulting equation, we get

$$\beta_m\beta_l\left(p.T^i_{jk}\right) = a_{lm}T^a_{bc}h^i_a h^b_j h^c_k\,.$$

Using (4.4) in above equation, we get

$$\text{(4.5)}\quad \beta_m\beta_l\left(p.T^i_{jk}\right) = a_{lm}\left(p.T^i_{jk}\right).$$

This shows that $p.T^i_{jk}$ is birecurrent. Thus, we conclude

Theorem 4.2: *The torsion tensor T^i_{jk} on indicatrix in pseudo $T-$birecurrent space is birecurrent in sense of Berwald.*

Let the projection of torsion tensor T^i_{jk} on indicatrix is birecurrent, i.e. characterized by (4.5). Using (1.3) in (4.5), we get

$$\beta_m\beta_l\left(T^a_{bc}h^i_a h^b_j h^c_k\right) = a_{lm}T^a_{bc}h^i_a h^b_j h^c_k\,.$$

Using (1.4) in above equation, we get

$$\beta_m\beta_l(T^i_{jk} - T^i_{bk}\ell^b\ell_j - T^a_{jk}\ell^i\ell_a + T^a_{bk}\ell^i\ell_a\ell^b\ell_j - T^i_{jc}\ell^c\ell_k + T^i_{bc}\ell^b\ell_j\ell^c\ell_k + T^a_{jc}\ell^i\ell_a + T^a_{bk}\ell^i\ell_a\ell^c\ell_k - T^i_{bc}\ell^i\ell_a\ell^b\ell_j\ell^c\ell_k)$$
$$= a_{lm}(T^i_{jk} - T^i_{bk}\ell^b\ell_j - T^a_{jk}\ell^i\ell_a + T^a_{bk}\ell^i\ell_a\ell^b\ell_j - T^i_{jc}\ell^c\ell_k + T^i_{bc}\ell^b\ell_j\ell^c\ell_k + T^a_{jc}\ell^i\ell_a + T^a_{bk}\ell^i\ell_a\ell^c\ell_k - T^i_{bc}\ell^i\ell_a\ell^b\ell_j\ell^c\ell_k)\,.$$

Using (2.1c), (1.2a) and (1.2c) in above equation, we get

$$\text{(4.6)}\quad \beta_m\beta_l(T^i_{jk} - \frac{1}{F}T^i_k\ell_j - T^a_{jk}\ell^i\ell_a + \frac{1}{F}T^a_k\ell^i\ell_a\ell_j - \frac{1}{F}T^i_j\ell_k + \frac{1}{F}T^a_c\ell_j\ell^c\ell_k + \frac{1}{F}T^a_j\ell^i\ell_a\ell_k - \frac{1}{F}T^a_c\ell^i\ell_a\ell_j\ell^c\ell_k)$$
$$a_{lm}(T^i_{jk} - T^i_{bk}\ell^b\ell_j - T^a_{jk}\ell^i\ell_a + T^a_{bk}\ell^i\ell_a\ell^b\ell_j - T^i_{jc}\ell^c\ell_k + T^i_{bc}\ell^b\ell_j\ell^c\ell_k + T^a_{jc}\ell^i\ell_a + T^a_{bk}\ell^i\ell_a\ell^c\ell_k - T^i_{bc}\ell^i\ell_a\ell^b\ell_j\ell^c\ell_k)$$

Now, since the division tensor T^i_j is birecurrent, i.e. satisfies (3.3). In view of (3.3), (1.2b) and (1.2c), then equation (4.6) can be written as

$$\beta_m\beta_l\left(T^i_{jk} - T^a_{jk}\ell^i\ell_a\right) = a_{lm}\left(T^i_{jk} - T^a_{jk}\ell^i\ell_a\right).$$

From last equation, we conclude

Corollary 4.2: *In pseudo $T-$birecurrent space, the projection of the torsion tensor T^i_{jk} on indicatrix is birecurrent, if and only if $T^a_{jk}\ell_a$ is birecurrent.*

We know that, the deviation tensor T^i_j is birecurrent, i.e. characterized by (3.3). In view of (1.3), the projection of the deviation tensor T^i_j on indicatrix is given by

(4.7) $p\ .T^i_j = T^a_b h^i_a h^b_j$.

Taking covariant derivative of (4.7) with respect to x^l and x^m in sense of Berwald and using the fact that $\beta_l h^i_j = 0$, then using (3.3) in the resulting equation, we get

$$\beta_m\beta_l\left(p.T^i_j\right) = a_{lm}T^a_b h^i_a h^b_j .$$

Using (4.7) in above equation, we get

(4.8) $\beta_m\beta_l\left(p.T^i_j\right) = a_{lm}\left(p.T^i_j\right)$.

This shows that $p.T^i_j$ is birecurrent. Thus, we conclude

Theorem 4.3: *The deviation tensor T^i_j on indicatrix pseudo $T-$birecurrent space is birecurrent in sense of Berwald.*

Let the projection of the division tensor T^i_j on indicatrix is birecurrent, i.e. characterized by (4.8). Using (1.3) in (4.8), we get

$$\beta_m\beta_l\left(T^a_b h^i_a h^b_j\right) = a_{lm}T^a_b h^i_a h^b_j .$$

Using (1.4) in above equation, we get

$$\beta_m\beta_l\left(T^i_j - T^i_b\ell^b\ell_j - T^a_j\ell^i\ell_a + T^a_b\ell^i\ell_a\ell^b\ell_j\right) = a_{lm}\left(T^i_j - T^i_b\ell^b\ell_j - T^a_j\ell^i\ell_a + T^a_b\ell^i\ell_a\ell^b\ell_j\right).$$

Now, in view of (1.2b), (1.2c) and if the division tensor $T^i_b y^b = 0$, then above equation becomes

$$\beta_m\beta_l\left(T^i_j - T^a_j\ell^i\ell_a\right) = a_{lm}\left(T^i_j - T^a_j\ell^i\ell_a\right).$$

From last equation, we conclude

Corollary 4.3: *In pseudo $T-$birecurrent space, the projection of the division tensor T^i_j on indicatrix is birecurrent, if and only if $T^a_j\ell_a$ is birecurrent.*

5. Conclusion

Some theorems belong to pseudo $T-$birecurrent space have been established and proved. Further, we discussed the projection on indicatrix for some tensors whose behave as birecurrent in sense of Berwald.

6. References

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