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An Example of an Elliptic Curve with High Rank Induced by Rational Diophantine Triples

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Abstract
In this paper, we give a certain example of an elliptic curve induced by rational Diophantine triples which has rank 4 following Andrej Dujella and Juan Carlos Peral's construction. After specifying the curve, we will write four infinite order independent rational points on the elliptic curve which shows that the rank is exactly four. Lastly, we will give some properties of the curve. Here we don't promise any new result and this work is a part of the first author's master thesis.

Keywords: Elliptic Curves, Rank, Torsion Subgroup, Diophantine Triples

1. Introduction

Elliptic curves have been a trend topic for decades. The connection between *Diophantine m-tuples* and elliptic curves is so interesting that there are many papers including recent ones, e.g. (Dujella and Peral, 2019) [2] and (Dujella and Peral, 2020) [3]. Let us consider the case where $\{a, b, c\}$ is a rational triple. Due to (Dujella and Peral, 2019) [2], then, there exist non-negative rationals r, s, t such that $ab + 1 = r^2$, $ac + 1 = s^2$ and $bc + 1 = t^2$. In order to extend the triple $\{a, b, c\}$ to a quadruple, one has to solve the system of equations.

$$ax + 1 = \square, bx + 1 = \square, cx + 1 = \square$$

It is a nice fact that we can attach an elliptic curve as

$$E: y^2 = (ax + 1)(bx + 1)(cx + 1)$$

for the system of equations above. Now we are ready to give the first definition of the paper.

Definition 1: Assume the set-up above. Then, we say that the elliptic curve E is induced by the rational Diophantine triple $\{a, b, c\}$.

With this interesting connection between *Diophantine triples* and elliptic curves, there are many research questions to ask. Before going further, first we should give the necessary background on elliptic curves, of course, the most classical reference for the topic is (Silverman, 1986) [4].

Due to (Silverman, 1986) [4], it is well-known that rational points on an elliptic curve form an abelian group with a special point addition law. It is possible to define the elliptic curves over different fields both global and local. In this paper, we stick on $\mathbb{Q}$. The theory of elliptic curves is really deep and we will take a part of it for this paper.

Definition 2: Let E be an elliptic curve over $\mathbb{Q}$. Then Mordell-Weil Theorem says that $E(\mathbb{Q})$ is finitely generated namely,

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \otimes \mathbb{Z}^r$$

where $E(\mathbb{Q})_{tors}$ is the finite torsion subgroup and ***r*** is defined as the rank of the elliptic curve.

An Example For an Elliptic Curve Which has Rank 4:
Roughly speaking, rank of an elliptic curve is a way to measure the size of rational points on an elliptic curve over $\mathbb{Q}$ and any work on this topic is indeed interesting. Note that a millennium problem, namely, Birch and Swinnerton-Dyer Conjecture has one part on rank of elliptic curves and it asserts "the analytical rank is equal to the algebraic rank".
Construction of infinite families of elliptic curves with high rank is an important problem and this problem connects the *Diophantine m-tuples* with the elliptic curves. In (Dujella and Peral, 2019) [2] give an explicit family of elliptic curves as the following:

$$y^2 = x^3 + A(a)x^2 + B(a)x$$

Where

$$A(a) = -2(-51200 + 109440a + 38880a^2 + 55404a^3 + 6561a^4),$$
$$B(a) = 243a^2(20 + 3a)(-4 + 9a)(16 + 9a)(80 + 9a)(320 + 81a^2)$$

In this work, specifying this family for the case $a = 4$, we have an example for an elliptic curve with rank 4 with the explicit proof and we give some properties of this curve. Here is the main theorem of the paper.

Theorem 3: Elliptic curve
$E: y^2 = x^3 - 12468224x^2 + 38808682758144x$
defined over $\mathbb{Q}$ has rank 4.
Proof: It is clear that the conductor of the elliptic curve is 840740160.
After a short calculation in Magma, we see that

G[1]:= (4632576, -3409575936)
G[2]:=(7817472, 4377784320)
G[3]:= (79872, -1738014720)
G[4]:= (20846592, 66709094400)

are rational points on the elliptic curve E which have order 4. It is also seen that they are independent points. This concludes the proof.

Theorem 4: Torsion subgroup of E is isomorphic $\mathbb{Z}_2 \otimes \mathbb{Z}_2$.
Proof: There are two rational points on the elliptic curve E with order 2, namely, $(6469632, 0)$ and $(5998592, 0)$. Recall that, due to (Silverman, 1986) [4], ordinates of the rational points of order 2 are zero. So, it is clear that torsion subgroup of the elliptic curve is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ and by Mordell-Weil theorem we have $E(\mathbb{Q}) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$.

2. Final Remarks
Magma Computer Algebra System (Bosma *et al*., 1997) [1] has very efficient algorithms especially for elliptic curves. One can use these algorithms for such problems. Note that the conductor of our example is out of range of John Cremona's database.

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