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On the non-homogenous quadratic with two unknowns $5x^2 - 2y^2 = 117$

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Abstract

The hyperbola represented by the binary quadratic equation $5x^2 - 2y^2 = 117$ is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

Keywords: Pell Like Equation, Binary Quadratic, Hyperbola, Parabola

1. Introduction

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-12].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $5x^2 - 2y^2 = 117$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

2. Method of analysis

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$5x^2 - 2y^2 = 117 \tag{1}$$

Introduction of the linear transformations

$$x = X + 2T, y = X + 5T \tag{2}$$

in (1) leads to

$$X^2 = 10T^2 + 39 \tag{3}$$

The smallest positive integer solution for (3) is $T_0 = 1, X_0 = 7$

To find the other solutions to (3), consider the corresponding pell equation given by

$$X^2 = 10T^2 + 1 \quad (4)$$

whose general solution $(\tilde{T}_n, \tilde{X}_n)$ is

$$\tilde{X}_n = \frac{1}{2} f_n$$
$$\tilde{T}_n = \frac{1}{2\sqrt{10}} g_n$$

Where,

$$f_n = (19 + 6\sqrt{10})^{n+1} + (19 - 6\sqrt{10})^{n+1}$$
$$g_n = (19 + 6\sqrt{10})^{n+1} - (19 - 6\sqrt{10})^{n+1}$$

- Employing the lemma of Brahmagupta between the solutions, $(T_0, X_0)\&(\tilde{T}_n, \tilde{X}_n)$ the general solution $(T_{n+1,} X_{n+1})$ to (3) is given by

$$T_{n+1} = T_0\tilde{X}_n + X_0\tilde{T}_n$$
$$= \frac{1}{2} f_n + 7 * \frac{1}{2\sqrt{10}} g_n$$
$$X_{n+1} = X_0\tilde{X}_n + DT_0\tilde{T}_n$$
$$= \frac{7}{2} f_n + \frac{\sqrt{10}}{2} g_n$$

where n= -2, -1, 0, 1,……
In view of (2), the general solution (x_{n+1}, y_{n+1}) to (1) is given by

$$x_{n+1} = X_{n+1} + 2T_{n+1}$$
$$= \frac{9}{2} f_n + 12 * \frac{1}{\sqrt{10}} g_n$$

$$y_{n+1} = X_{n+1} + 5T_{n+1}$$
$$= 6f_n + 45 * \frac{1}{2\sqrt{10}} g_n$$

The recurrence relation satisfied by x and y are given by

$$x_{n+1} - 38x_{n+2} + x_{n+3} = 0$$
$$y_{n+1} - 38y_{n+2} + y_{n+3} = 0$$

A few numerical solutions to (1) are presented in table below:

Table 1: Numerical solutions

n	x_{n+1}	y_{n+1}
-2	27	-42
-1	9	12
0	315	498
1	11961	18912
2	454203	718158
3	17247753	27271092

From the above table, we observe some interesting relations among the solutions which are presented below:
1) x_{n+1} are always even and y_{n+1} are always odd.

2) Each of the following expressions is a nasty number
$(i) 517920x_{2n+2} - 12480x_{2n+3} - 730080$
$(ii) 747402240x_{2n+2} - 474240x_{2n+4} - 1054235520$
$(iii) 70200x_{2n+2} - 37440y_{2n+2} - 182520$
$(iv) 46683000x_{2n+2} - 711360y_{2n+3} - 65889720$

$(v) 269065087000x_{2n+2} - 107976960y_{2n+4} - 3795255190000$
$(vi) 9692705620000x_{2n+3} - 255233048000x_{2n+4} - 359786344000$
$(vii) 48016800x_{2n+3} - 1062771840y_{2n+4} - 2372029920$
$(viii) 9828000x_{2n+3} - 6215040y_{2n+3} - 730080$
$(ix) 7090480800x_{2n+3} - 118085760y_{2n+4} - 263558880$
$(x) 50614200x_{2n+4} - 42542922240y_{2n+2} - 94881379320$
$(xi) 186732000x_{2n+4} - 4484413440y_{2n+3} - 263558880$
$(xii) 93295800x_{2n+4} - 59005440y_{2n+4} - 182520$
$(xiii) 526500y_{2n+3} - 18427500y_{2n+2} - 41067000$
$(xiv) 10985463570000y_{2n+4} - 1459968109000000y_{2n+2} - 32560913980000000$
$(xv) 112112900000y_{2n+4} - 4257087354000y_{2n+3} - 249851628000$

3. Remarkable observations

1). Employing linear combinations among the solution of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table below:

Table 2: Observations

S.NO	Hyperbola	x, y
1.	$16x^2 - 90y^2 = 219024$	$x = 83x_{n+1} - 2x_{n+2}$ $y = x_{n+2} - 35x_{n+1}$
2.	$64x^2 - 90y^2 = 316270656$	$x = 1576x_{n+1} - x_{n+3}$ $y = x_{n+3} - 1329x_{n+1}$
3.	$4x^2 - 10y^2 = 6084$	$x = 15x_{n+1} - 8y_{n+1}$ $y = 6y_{n+1} - 8x_{n+1}$
4.	$4x^2 - 10y^2 = 2196324$	$x = 525x_{n+1} - 8y_{n+2}$ $y = 6y_{n+2} - 332x_{n+1}$
5.	$4x^2 - 40y^2 = 3162712644$	$x = 19935x_{n+1} - 8y_{n+3}$ $y = 3y_{n+3} - 6304x_{n+1}$
6.	$x^2 - 810y^2 = 1971216$	$x = 37824x_{n+2} - 996x_{n+3}$ $y = 35x_{n+3} - 1329x_{n+2}$
7.	$4x^2 - 40y^2 = 19766916$	$x = 45x_{n+2} - 996y_{n+1}$ $y = 315y_{n+1} - 12x_{n+2}$
8.	$4x^2 - 360y^2 = 54756$	$x = 1575x_{n+2} - 996y_{n+2}$ $y = 105y_{n+2} - 166x_{n+2}$
9.	$4x^2 - 40y^2 = 2196324$	$x = 19935x_{n+2} - 332y_{n+3}$ $y = 105y_{n+3} - 6304x_{n+2}$
10.	$4x^2 - 10y^2 = 3162712644$	$x = 15x_{n+3} - 12608y_{n+1}$ $y = 7974y_{n+1} - 8x_{n+3}$
11.	$4x^2 - 40y^2 = 2196324$	$x = 525x_{n+3} - 12608y_{n+2}$ $y = 3987y_{n+2} - 166x_{n+3}$
12.	$x^2 - 10y^2 = 6084$	$x = 39870x_{n+3} - 25216y_{n+3}$ $y = 7974y_{n+3} - 12608x_{n+3}$
13.	$25x^2 - 10y^2 = 1368900$	$x = 3y_{n+2} - 105y_{n+1}$ $y = 166y_{n+1} - 4y_{n+2}$
14.	$123543225x^2 - 21963240y^2 = 1085363800000000$	$x = y_{n+3} - 1329y_{n+1}$ $y = 3152y_{n+1} - 2y_{n+3}$
15.	$342225x^2 - 60840y^2 = 8328387600$	$x = 70y_{n+3} - 2658y_{n+2}$ $y = 6304y_{n+2} - 166y_{n+3}$

2). Employing linear combinations among the solution of (1), one may generate integer solutions for other choices of parabola which are presented in the table below:

Table 3: Observations

S. No	Parabola	x, y
1.	$104x - 10y^2 = 24336$	$x = 83x_{2n+2} - 2x_{2n+3} + 117$ $y = x_{n+2} - 35x_{n+1}$
2.	$7904x - 10y^2 = 35141184$	$x = 1576x_{2n+2} - x_{2n+4} + 2223$ $y = x_{n+3} - 1329x_{n+1}$
3.	$78x - 10y^2 = 6084$	$x = 15x_{2n+2} - 8y_{2n+2} + 39$ $y = 6y_{n+1} - 8x_{n+1}$
4.	$1482x - 10y^2 = 2196324$	$x = 525x_{2n+2} - 8y_{2n+3} + 741$ $y = 6y_{n+2} - 332x_{n+1}$

5.	$56238x - 40y^2 = 3162712640$	$x = 19935x_{2n+2} - 8y_{2n+4} + 28119$ $y = 3y_{n+3} - 6304x_{n+1}$
6.	$6084x - 7020y^2 = 17083872$	$x = 37824x_{2n+3} - 996x_{2n+4} + 1404$ $y = 35x_{n+3} - 1329x_{n+2}$
7.	$4446x - 40y^2 = 19766916$	$x = 45x_{2n+3} - 996y_{2n+2} + 2223$ $y = 315y_{n+1} - 12x_{n+2}$
8.	$26x - 40y^2 = 6084$	$x = 1575x_{2n+3} - 996y_{2n+3} + 117$ $y = 105y_{n+2} - 166x_{n+2}$
9.	$1482x - 40y^2 = 2196324$	$x = 19935x_{2n+3} - 332y_{2n+4} + 741$ $y = 105y_{n+3} - 6304x_{n+2}$
10.	$56238x - 10y^2 = 3162712644$	$x = 15x_{2n+4} - 12608y_{2n+2} + 28119$ $y = 7974y_{n+1} - 8x_{n+3}$
11.	$1482x - 40y^2 = 2196324$	$x = 525x_{2n+4} - 12608y_{2n+3} + 741$ $y = 3987y_{n+2} - 166x_{n+3}$
12.	$39x - 10y^2 = 6048$	$x = 39870x_{2n+4} - 25216y_{2n+4} + 78$ $y = 7974y_{n+3} - 12608x_{n+3}$
13.	$2925x - 10y^2 = 1368900$	$x = 3y_{2n+3} - 105y_{2n+2} + 234$ $y = 166y_{n+1} - 4y_{n+2}$
14.	$123543225x - 14820y^2 = 732364237800$	$x = y_{2n+4} - 1329y_{2n+2} + 2964$ $y = 3152y_{n+1} - 2y_{n+3}$
15.	$342225x - 780y^2 = 106774200$	$x = 70y_{2n+4} - 2658y_{2n+3} + 156$ $y = 6304y_{n+2} - 166y_{n+3}$

4. Conclusion

In this paper, we have made an attempt to obtain all integer solutions through a single process. To conclude, one may search for the equations for which the above method is applicable.

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