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On the Positive Pell Equation $y^2 = 14x^2 + 18$

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Abstract

The binary quadratic equation $y^2 = 14x^2 + 18$ representing the hyperbola is studied for its non-zero distinct integer solutions. A few interesting properties among the solutions are presented. Employing the integer solutions of the equation under consideration, integer solutions for special straight lines, hyperbolas and parabolas are exhibited.

Keywords: Binary Quadratic, Hyperbola, Parabola, Integer Solutions, Pell Equation

1. Introduction

The binary quadratic equation of the form $y^2 = \mathrm{D}x^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-11]. In this communication, yet another interesting hyperbola given by $y^2 = 14x^2 + 18$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. Method of analysis

The positive pell equation representing hyperbola under consideration is

$$y^2 = 14x^2 + 18 \tag{1}$$

whose smallest positive integer solutions is

$$x_0 = 3, y_0 = 12$$

To obtain the other solutions of (1), Consider the pell equation

$$y^2 = 14x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{14}} g_n$$
$$\tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (15+4\sqrt{14})^{n+1} + (15-4\sqrt{14})^{n+1}$$
$$g_n = (15+4\sqrt{14})^{n+1} - (15-4\sqrt{14})^{n+1}$$

Appling Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{3}{2} f_n + \frac{6}{\sqrt{14}} g_n$$
$$y_{n+1} = 6 f_n + \frac{3\sqrt{14}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 30x_{n+2} + x_{n+3} = 0$$
$$y_{n+1} - 30y_{n+2} + y_{n+3} = 0$$

Some numerical examples of the x and y satisfying (1) are given in the following table

n	x_{n+1}	y_{n+1}
-1	3	12
0	93	348
1	2787	10428
2	83517	312492
3	2502723	9364332

From the above table, we observe some interesting relations among the solutions which are presented below.

- x_{n+1} are always odd and y_{n+1} are always even.
- Each of the following expressions is a nasty number

  - $1008x_{2n+3} - 29232x_{2n+2} - 6048$
  - $30240x_{2n+4} - 26278560x_{2n+2} - 907200$
  - $252y_{2n+2} - 882x_{2n+2} - 126$
  - $15120y_{2n+3} - 1640520x_{2n+2} - 340200$
  - $452592y_{2n+4} - 1471602888x_{2n+2} - 304820712$
  - $116928x_{2n+4} - 3503808x_{2n+3} - 24192$
  - $438480y_{2n+2} - 5292x_{2n+3} - 340200$
  - $29232y_{2n+3} - 109368x_{2n+3} - 1512$
  - $438480y_{2n+4} - 8193780x_{2n+3} - 56700$
  - $393302448y_{2n+2} - 1584072x_{2n+4} - 304820712$
  - $13139280y_{2n+3} - 1640520x_{2n+4} - 340200$
  - $281232y_{2n+2} - 9072y_{2n+3} - 36288$
  - $875952y_{2n+4} - 3277512x_{2n+4} - 1512$
  - $1755810y_{2n+2} - 1890y_{2n+2} - 1360800$
  - $936432y_{2n+3} - 31248y_{2n+4} - 24192$

3. Remarkable observations

Table 1: Employing linear combinations among the solution of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table below

S. No	Hyperbola	x, y
1.	$8x^2 - 7y^2 = 288$	$x = x_{n+2} - 29x_{n+1}$ $y = 31x_{n+2} - x_{n+2}$
2.	$8x^2 - 7y^2 = 259200$	$x = x_{n+3} - 869x_{n+1}$ $y = 929x_{n+1} - x_{n+3}$
3.	$2x^2 - 7y^2 = 18$	$x = 2y_{n+1} - 7x_{n+1}$ $y = 4x_{n+1} - y_{n+1}$
4.	$2x^2 - 7y^2 = 4050$	$x = 2y_{n+2} - 217x_{n+1}$ $y = 116x_{n+1} - y_{n+2}$
5.	$x^2 - 14y^2 = 7257636$	$x = 4y_{n+3} - 13006x_{n+1}$ $y = 3476x_{n+1} - y_{n+3}$
6.	$144x^2 - 1134y^2 = 46656$	$x = 87x_{n+3} - 2607x_{n+2}$ $y = 929x_{n+2} - 31x_{n+3}$
7.	$x^2 - 14y^2 = 8100$	$x = 116y_{n+1} - 14x_{n+2}$ $y = 4x_{n+2} - 31y_{n+1}$
8.	$x^2 - 14y^2 = 36$	$x = 116y_{n+2} - 434x_{n+2}$ $y = 116x_{n+2} - 31y_{n+2}$
9.	$x^2 - 14y^2 = 8100$	$x = 116y_{n+3} - 13006x_{n+2}$ $y = 3476x_{n+2} - 31y_{n+3}$
10.	$x^2 - 14y^2 = 7257636$	$x = 3476y_{n+1} - 14x_{n+3}$ $y = 4x_{n+3} - 929y_{n+1}$
11.	$x^2 - 14y^2 = 8100$	$x = 3476y_{n+2} - 434x_{n+3}$ $y = 116x_{n+3} - 929y_{n+2}$
12.	$126x^2 - 144y^2 = 72576$	$x = 31y_{n+1} - y_{n+2}$ $y = y_{n+2} - 29y_{n+1}$
13.	$x^2 - 14y^2 = 36$	$x = 3476y_{n+3} - 13006x_{n+3}$ $y = 3476x_{n+3} - 929y_{n+3}$
14.	$567x^2 - 648y^2 = 293932800$	$x = 929y_{n+1} - y_{n+3}$ $y = y_{n+3} - 869y_{n+1}$
15.	$14x^2 - y^2 = 8064$	$x = 929y_{n+2} - 31y_{n+3}$ $y = 116y_{n+3} - 3476y_{n+2}$

Table 2: Employing linear combinations among the solution of (1), one may generate integer solutions for other choices of parabola which are presented in the table below

S. No	Parabola	x, y
1.	$24x - 7y^2 = 288$	$x = x_{2n+3} - 29x_{2n+2} + 6$ $y = 31x_{n+2} - x_{n+2}$
2.	$720x - 7y^2 = 259200$	$x = x_{2n+4} - 869x_{2n+2} + 180$ $y = 929x_{n+1} - x_{n+3}$
3.	$3x - 7y^2 = 18$	$x = 2y_{2n+2} - 7x_{2n+2} + 3$ $y = 4x_{n+1} - y_{n+1}$
4.	$90x - 14y^2 = 8100$	$x = 2y_{2n+3} - 217x_{2n+2} + 45$ $y = 116x_{n+1} - y_{n+2}$
5.	$1347x - 14y^2 = 7257636$	$x = 4y_{2n+4} - 13006x_{2n+2} + 2694$ $y = 3476x_{n+1} - y_{n+3}$
6.	$16x - 14y^2 = 576$	$x = 87x_{2n+4} - 2607x_{2n+3} + 18$ $y = 929x_{n+2} - 31x_{n+3}$
7.	$45x - 14y^2 = 8100$	$x = 116y_{2n+2} - 14x_{2n+3} + 90$ $y = 4x_{n+2} - 31y_{n+1}$
8.	$3x - 14y^2 = 36$	$x = 116y_{2n+3} - 434x_{2n+3} + 6$ $y = 116x_{n+2} - 31y_{n+2}$
9.	$45x - 14y^2 = 8100$	$x = 116y_{2n+4} - 13006x_{2n+3} + 90$ $y = 3476x_{n+2} - 31y_{n+3}$
10.	$1347x - 14y^2 = 7257636$	$x = 3476y_{2n+2} - 14x_{2n+4} + 2694$ $y = 4x_{n+3} - 929y_{n+1}$
11.	$45x - 14y^2 = 8100$	$x = 3476y_{2n+3} - 434x_{2n+4} + 90$ $y = 116x_{n+3} - 929y_{n+2}$
12.	$126x - 12y^2 = 6048$	$x = 31y_{2n+2} - y_{2n+3} + 24$ $y = y_{n+2} - 29y_{n+1}$
13.	$3x - 14y^2 = 36$	$x = 3476y_{2n+4} - 13006x_{2n+4} + 6$ $y = 3476x_{n+3} - 929y_{n+3}$
14.	$315x - y^2 = 453600$	$x = 929y_{2n+2} - y_{2n+4} + 720$ $y = y_{n+3} - 869y_{n+1}$
15.	$168x^2 - y^2 = 8064$	$x = 929y_{2n+3} - 31y_{2n+4} + 24$ $y = 116y_{n+3} - 3476y_{n+2}$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive pell equation $y^2 = 14x^2 + 18$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of positive pell equations and determine their solutions along with suitable properties.

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