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Exact symmetry reduction solutions of a nonlinear coupled system of Korteweg-De Vries Equations

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Abstract

We study a system of coupled Kortewegde Vries equations that model the propagation of shallow water waves, ion-acoustic waves in plasmas, solitons, and nonlinear perturbations along internal surfaces between layers of different densities in stratified fluids, for example propagation of solitons of long internal waves in oceans. Other applications of this kind of equations have been to model shock wave formation, turbulence, boundary layer behavior, and mass transport. The method presented is Lie group analysis. We first obtain Lie point symmetries and use them to carry out symmetry reductions and the resulting systems investigated for solutions. Traveling waves are constructed by use of a linear combination of time and space translation symmetries.

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Introduction

The dynamics of shallow-water waves, ion-acoustic waves in plasmas, and long internal waves in oceans can be described by coupled KdV equations. The equations are derived from the classical kdv equation. This section extends the previous study of kdV equations to that of a coupled nonlinear system. From the Kortweg-de Vries equation

$$H_t + \alpha H H_x + \beta H_{xxx} = 0, \tag{1}$$

for α and β as constants, we let

$$H(t, x) = u(t, x) + iv(t, x), \tag{2}$$

where $i^2 = -1$. Then substituting (2) into (1) and separating the real and imaginary parts, we obtain

$$\Delta_1 \equiv u_t + \alpha u u_x - \alpha v v_x + \beta u_{xxx} = 0, \tag{3}$$

$$\Delta_2 \equiv v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx} = 0, \tag{4}$$

which is a nonlinear system of coupled KdV equations. We perform Lie symmetry analysis on (3), that is , we obtain Lie point symmetries, invariant solutions and conservation laws of (3). This paper uses Lie group analysis method to construct exact solutions and conservation laws for a nonlinear coupled kdV system (3).

Preliminaries

In this section, we outline preliminary concepts which are useful in the sequel.

Local Lie groups. [5] In Euclidean spaces R^n of $x = x^i$ independent variables and R^m of $u = u^\alpha$ dependent variables, we consider the transformations

$$T_\epsilon : \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{5}$$

involving the continuous parameter ϵ which ranges from a neighbourhood $N' \subset N \subset R$ of $\epsilon = 0$ where the functions φ^i and ψ^α differentiable and analytic in the parameter ϵ

Definition 0.1. The set of transformations given by (5) is a local Lie group if it holds true that

1. (Closure) Given $T_{\epsilon 1}, T_{\epsilon 2} \in G$, for $\epsilon_1, \epsilon_2 \in N' \subset N$, then $T_{\epsilon 1}\, T_{\epsilon 2} = T_{\epsilon 3} \in G$, $\epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in N$.

2. (Identity) There exists a unique $T_0 \in G$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$.

3. (Inverse) There exists a unique $T_{\epsilon-1} \in G$ for every transformation $T_\epsilon \in G$, Where $\epsilon \in N' \subset N$ and $\epsilon^{-1} \in N$ such that $T_\epsilon T_{\epsilon-1} = T_{\epsilon-1}\, T_\epsilon = T_0$.

Remark 0.2. Associativity of the group G in (5) follows from (1).
Prolongations. In the system,

$$\Delta_\alpha\, x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)} = \Delta_\alpha = 0, \tag{6}$$

the variables u^α are dependent. The partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u^\alpha\ \}, \ldots, u_{(\pi)} = \{u^\alpha{}_{i1\ldots i\pi}\}$, are of the first, second, . . . , up to the πth-orders.

Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \tag{7}$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta, we have

$$D_i(x^j) = \delta_i^j, ', \ u_i^\alpha = D_i(u^\alpha), \ \ u_{ij}^\alpha = D_j(D_i(u^\alpha)), \ \ \ldots, \tag{8}$$

where u_i^α defined in (8) are differential variables [8].

(1) Prolonged groups Consider the local Lie group $\mathcal{G}$ given by the transformations

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \ \ \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{9}$$

where the symbol $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 0.3. The construction of the group $\mathcal{G}$ given by (9) is an equivalence of the computation of infinitesimal transformation

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ \bar{u}^\alpha &\approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{10}$$

obtained from (5) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} \tag{11}$$

Remark 0.4. The symbol of infinitesimal transformations, X, is used to write (10) as

$$\bar{x}^i \approx (1 + X)x^i, \bar{u}^\alpha \approx (1 + X)u^\alpha, \tag{12}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{13}$$

is the generator of the group $\mathcal{G}$ given by (9).

Remark 0.5. To obtain transformed derivatives from (5), we use a change of variable formulae

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{14}$$

where $D^{-}j$ is the total differentiation in the variables x^{-i}. This means that

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha),\ \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha) \tag{15}$$

If we apply the change of variable formula given in (14) on $\mathcal{G}$ given by (9), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j),\ \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j) \tag{16}$$

Expansion of (16) yields

$$\left(\frac{\partial\varphi^j}{\partial x^i} + u_i^\beta\frac{\partial\varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial\psi^\alpha}{\partial x^i} + u_i^\beta\frac{\partial\psi^\alpha}{\partial u^\beta} \tag{17}$$

The variables $^{-}u^\alpha_i$ can be written as functions of x^i, u^α, $u(1)$, that is

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha \tag{18}$$

Definition 0.6. The transformations in the space of the vari- ables x^i, u^α, $u_{(1)}$ given in (9) and (18) form the first prolongation group $\mathcal{G}^{[1]}$.

Definition 0.7. Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha\epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon) \tag{19}$$

Remark 0.8. In terms of infinitesimal transformations, the first prolongation group $\mathcal{G}^{[1]}$ is given by (10) and (19).

(2) Prolonged generators

Definition 0.9. By using the relation given in (16) on the first prolongation group $\mathcal{G}^{[1]}$ given by Definition 0.6, we obtain [5]

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \quad \text{which gives} \tag{20}$$

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{21}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{22}$$

is the first prolongation formula.

Remark 0.10. Similarly, we get higher order prolongations [9],

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \quad \ldots, \quad \zeta_{i_1,\ldots,i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j) \tag{23}$$

Remark 0.11. The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \ldots, \mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1,\ldots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\ldots,i_\kappa}^\alpha}, \quad \kappa \geq 1, \tag{24}$$

where X is the group generator given by (13).

Group invariants.

Definition 0.12. A function $\Gamma(x^i, u^\alpha)$ is called an invariant of the group $\mathcal{G}$ of transformations given by (5) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha) \tag{25}$$

Theorem 0.13. *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (5) if and only if it solves the following first-order linear PDE:* [5]

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0 \tag{26}$$

From Theorem (0.13), we have the following result.

Theorem 0.14. *The local Lie group $\mathcal{G}$ of transformations in R^n given by (5)* [8] *has precisely $n - 1$ functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{27}$$

Of the characteristic equations for (26):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)} \tag{28}$$

Symmetry groups.

Definition 0.15. The vector field X (13) is a Lie point symmetry of the PDE system (6) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \tag{29}$$

are satisfied, where $\Big|_{\Delta_\alpha=0}$ means evaluated on $\Delta_\alpha = 0$ and $X^{[\pi]}$ is the π-th prolongation of X.

Definition 0.16. The Lie group $\mathcal{G}$ is a symmetry group of the PDE system given in (6) if the PDE system (6) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0 \tag{30}$$

Theorem 0.17. *Given the infinitesimal transformations in (9), the Lie group $\mathcal{G}$ in (5) is found by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{31}$$

Lie algebras.

Definition 0.18. A vector space V_r of operators [5] X (13) is a Lie algebra if for any two operators, $X_i, X_j \in V_r$, their commutator

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{32}$$

is in V_r for all $i, j = 1, \ldots, r$.

Remark 0.19. The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity [5].

Theorem 0.20. *The set of solutions of the determining equation given by (29) forms a Lie algebra* [5]. [21]

Exact solutions. The methods of (G'/G)-expansion method, Extended Jacobi elliptic function expansion [22] and Kudryashov [23] are usually applied after symmetry reductions.

Conservation laws. [10]

Fundamental operators. Let a system of πth-order PDEs be given by (6).

Definition 0.21. The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa>1}(-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{33}$$

and the Lie- B¨acklund operator in abbreviated form [5] is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \ldots. \tag{34}$$

Remark 0.22. The Lie- B¨acklund operator (34) in its prolonged form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1} \zeta_{i_1 \ldots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{35}$$

Where

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \ldots, \zeta^\alpha_{i_1 \ldots i_\kappa} = D_{i_1 \ldots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_\kappa}, \quad j = 1, \ldots, n \tag{36}$$

and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j \tag{37}$$

Remark 0.23. The characteristic form of Lie- B¨acklund operator (35) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \ldots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}} \tag{38}$$

Remark 0.24. Noether's Theorem is applicable to systems from variational problems

The method of multipliers.

Definition 0.25. A function $\Lambda^\alpha(x^i, u^\alpha, u_{(1)}, \ldots) = \Lambda^\alpha$, is a multiplier of the PDE system given by (6) if it satisfies the condition that [17]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{39}$$

where $D_i T^i$ is a divergence expression

Definition 0.26. To find the multipliers Λ^α, one solves the determining equations (40) [3],

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0 \tag{40}$$

Ibragimov's conservation theorem. The technique [10] enables one to construct conserved vectors associated with each Lie point symmetry of the PDE system given by (6).

Definition 0.27. The adjoint equations of the system given by (6) are

$$\Delta^*_\alpha\left(x^i, u^\alpha, v^\alpha, \ldots, u_{(\pi)}, v_{(\pi)}\right) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta \Delta_\beta) = 0, \tag{41}$$

where v^α is the new dependent variable.

Definition 0.28. Formal Lagrangian $\mathcal{L}$ of the system (6) and its adjoint equations (41) is [10]

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}) \tag{42}$$

Theorem 0.29. *Every infinitesimal symmetry X of the system given by (6) leads to conservation laws* [10]

$$D_i T^i\Big|_{\Delta_\alpha=0} = 0, \tag{43}$$

where the conserved vector

$$T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] +$$
$$D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right]. \tag{44}$$

Main results

An illustrative example with a simple kdV equation can be found in [7].We now present our results in this section.

Nonlinear Coupled Korteweg-de Vries (KdV) Equations.

Lie point symmetries and solutions of the nonlinear coupled KdV Equations (3). The infinitesimal transformations of the Lie group with parameter ϵ are

$$\begin{aligned} \bar{t} &= t + \xi^t(t,x,u,v)\epsilon, \ \ \bar{x} = x + \xi^x(t,x,u,v)\epsilon, \\ \bar{u} &= u + \eta^u(t,x,u,v)\epsilon, \ \ \bar{v} = v + \eta^v(t,x,u,v)\epsilon \end{aligned} \tag{45}$$

The vector field

$$X = \xi^t(t,x,u,v)\frac{\partial}{\partial t} + \xi^x(t,x,u,v)\frac{\partial}{\partial x} + \eta^u(t,x,u,v)\frac{\partial}{\partial u} + \eta^v(t,x,u,v)\frac{\partial}{\partial v}, \tag{46}$$

is a Lie point symmetry of (3) if

$$\mathrm{X}^{[3]}\Delta_1\Big|_{\Delta_1=0,\ \Delta_2=0} = 0, \quad \mathrm{X}^{[3]}\Delta_2\Big|_{\Delta_1=0,\ \Delta_2=0} = 0 \tag{47}$$

Expanding (47) and and splitting on derivatives of v and u, we have an overdetermined system of ten PDEs, namely,

$$\begin{aligned} \xi^t_u = 0, \ \xi^t_v = 0, \ \xi^t_x = 0, \ \xi^x_u = 0, \ \xi^x_v = 0, \ \xi^t_{tt} = 0, \\ \xi^x_{tt} = 0, \ 3\xi^x_x - \xi^t_t = 0, \\ 3\eta^v + 2\xi^t_t v = 0, \ 3\alpha\eta^u + 2\alpha\xi^t_t u - 3\xi^x_t = 0. \end{aligned} \tag{48}$$

Solving the system (48) yields

$$\xi^t = A_1 + 3A_2 t, \tag{49}$$

$$\xi^x = A2x + \alpha A_3 t + A_4, \ \eta^u = -2A_2 u + A_3, \ \ \eta^v = -2A_2 v, \tag{50}$$

for arbitrary constants *A1, A2, A3, A4*. Hence from (49), the infinites imal symmetries of the coupled KdV Equations (3) is a Lie algebra generated by the vector fields

$$X_1 = \frac{\partial}{\partial t}, \ \ X_2 = \frac{\partial}{\partial x}, \ \ X_3 = \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{51}$$

$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} \tag{52}$$

0.0.1. *Commutator table*. The set of all infinitesimal symmetries of coupled KdV equations forms a Lie algebra and yield the following commutation relations in Table 1.

Table 1: Commutation relations

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	αX_2	$3X_1$
X_2	0	0	0	X_2
X_3	$-\alpha X_2$	0	0	$-2X_3$
X_4	$-3X_1$	$-X_2$	$2X_3$	0

A commutator table for the Lie algebra generated by the symmetries of coupled KdV equation

0.0.2. *Local Lie groups*. The following Lie groups, for $i = 1, 2, 3, 4$, are obtained

$$T_{\epsilon_1} : \bar{t} = t + \epsilon_1,\ \bar{x} = x,\ \bar{u} = u,\ \bar{v} = v, \tag{53}$$

$$T_{\epsilon_2} : \bar{t} = t,\ \bar{x} = x + \epsilon_2,\ \bar{u} = u,\ \bar{v} = v, \tag{54}$$

$$T_{\epsilon_3} : \bar{t} = t,\ \bar{x} = x + \alpha\epsilon_3 t,\ \bar{u} = u + \epsilon_3,\ \bar{v} = v, \tag{55}$$

$$T_{\epsilon_4} : \bar{t} = te^{3\epsilon_4},\ \bar{x} = xe^{\epsilon_4},\ \bar{u} = ue^{-2\epsilon_4},\ \bar{v} = ve^{-2\epsilon_4} \tag{56}$$

Symmetry reductions of the coupled KdV Equations (3). The symme tries obtained yield the following symmetry reductions.

(i) **The time translation symmetry**

$$X_1 = \frac{\partial}{\partial t} \tag{57}$$

Solving the characteristic equations

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0}, \tag{58}$$

associated to the operator X1 gives the invariants

$$J_1 = x,\ \ J_2 = u,\ \ J_3 = v \tag{59}$$

Hence, we have

$$u = \varphi(x),\ \ v = \psi(x), \tag{60}$$

for arbitrary functions φ and ψ. Substituting the expressions for u and v given by (60) into the system (3), we get a system of third order ordinary DEs namely,

$$\alpha\left[\varphi(x)\varphi'(x) - \psi(x)\psi'(x)\right] + \beta\varphi'''(x) = 0,\ \ \alpha\left(\varphi(x)\psi(x)\right)' + \beta\psi'''(x) = 0 \tag{61}$$

Integration of the system (??)-(61) yields;

$$\frac{\alpha}{2}\left[\varphi(x)^2 - \psi(x)^2\right] + \beta\varphi''(x) = C_1, \tag{62}$$

$$\alpha\left[\varphi(x)\psi(x)\right] + \beta\psi''(x) = C_2, \tag{63}$$

for arbitrary constants C1 and C2. If we take

$$C_1 = C_2 = 0, \tag{64}$$

the system (62)-(63) becomes

$$\frac{\alpha}{2}\left[\varphi(x)^2 - \psi(x)^2\right] + \beta\varphi''(x) = 0, \tag{65}$$

$$\alpha\left[\varphi(x)\psi(x)\right] + \beta\psi''(x) = 0 \tag{66}$$

To find more solutions of the system (65)-(66), we determine its Lie point symmetries. Using the Lie's algorithm for computing point symmetries, we see that the Lie point symmetries of (65)- (66) are

$$X_1^* = \frac{\partial}{\partial x},\ \ X_2^* = x\frac{\partial}{\partial x} - 2\varphi\frac{\partial}{\partial \varphi} - 2\psi\frac{\partial}{\partial \psi} \tag{67}$$

Proceeding as above, we see that the symmetry $X^*{}_1$ yields the trivial solution

$$u = 0, v = 0 \tag{68}$$

The second symmetry $X^*{}_2$ has the characteristic equations

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}\varphi}{-2\varphi} = \frac{\mathrm{d}\psi}{-2\psi}, \tag{69}$$

which provides the invariants

$$J_1 = x^2\varphi, \quad J_2 = x^2\psi \tag{70}$$

Letting

$$\varphi = \frac{\lambda}{x^2}, \quad \psi = \frac{\mu}{x^2}, \tag{71}$$

substituting the values of φ and ψ into (65)-(66) and solving the resulting equations yield.

(a) Case one. Taking

$$\mu = 0 \tag{72}$$

gives

$$\lambda = 0 \tag{73}$$

or

$$\lambda = -\frac{12\beta}{\alpha} \tag{74}$$

When

$$\lambda = 0, \text{ and } \quad \mu = 0, \tag{75}$$

we also get the trivial solution (68). One can easily see that if

$$\lambda = -\frac{12\beta}{\alpha}, \text{ and } \mu = 0, \tag{76}$$

then

$$\varphi = -\frac{12\beta}{\alpha x^2}, \quad \psi = 0, \tag{77}$$

which is a solution of the system (65)-(66). Hence

$$u_1(t,x) = -\frac{12\beta}{\alpha x^2}, \quad v_1(t,x) = 0, \tag{78}$$

is a solution of the coupled KdV system (3).

(b) **Case two**. Taking

$$\lambda = -\frac{6\beta}{\alpha} \tag{79}$$

gives

$$\mu = \pm\frac{6\beta i}{\alpha}, \tag{80}$$

with i 2 = −1. Consequently,

$$u_2(t,x) = -\frac{6\beta}{\alpha x^2}, \quad v_2(t,x) = \frac{6i\beta}{\alpha x^2}, \tag{81}$$

and

$$u_3(t,x) = -\frac{6\beta}{\alpha x^2}, \;\; v_3(t,x) = -\frac{6i\beta}{\alpha x^2}, \tag{82}$$

are solutions of the coupled KdV system (3). Hence Lie group analysis has given us three steady-state solutions for the coupled KdV system (3) under the time translation symmetry X1 = ∂/∂t.

(ii) **The space translation symmetry**

$$X_2 = \frac{\partial}{\partial x} \tag{83}$$

Solving the characteristic equations

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0}, \tag{84}$$

associated to X2 gives the invariants

$$J_1 = t, \;\; , \; J_2 = u \;\; J_3 = v \tag{85}$$

Therefore, the group-invariant solution is

$$u = \phi(t), \;\; v = h(t) \tag{86}$$

for arbitrary functions h and ϕ. Substitution of the solutions from (86) into (3), we get a system of first order ordinary DEs, namely,

$$\phi'(t) = 0, \;\; h'(t) = 0, \tag{87}$$

which is integrated once with respect to t to yield

$$\phi(t) = C_1, \;\; h(t) = C_2, \tag{88}$$

for arbitrary constants C_1 and C_2. Consequently, the space translation group-invariant solution of the system (3) is

$$u(t,x) = C_1, \;\; v(t,x) = C_2 \tag{89}$$

(iii) **The Galilean boost symmetry**

$$X_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{90}$$

Solving the characteristic equations

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\alpha t} = \frac{\mathrm{d}u}{1} = \frac{\mathrm{d}v}{0}, \tag{91}$$

associated to Galilean boost gives the invariants

$$J_1 = t, \;\; J_2 = v, \;\; J_3 = -u + \frac{x}{\alpha t}, \; t \neq 0 \tag{92}$$

Thus the invariant solution of (3) is

$$u = \frac{x}{\alpha t} - g(t), \;\; v = f(t), \; t \neq 0, \tag{93}$$

for arbitrary functions f and g. Substitution of the values of u and v from (93) into the System (3), we get a nonlinear system of coupled first order ordinary DEs, namely,

$$tg'(t) + g(t) = 0, \;\; tf'(t) + f(t) = 0, \tag{94}$$

whose solutions are

$$g(t) = \frac{C_1}{t} \;\; f(t) = \frac{C_2}{t}, \tag{95}$$

for arbitrary constants C_1 and C_2. Hence the Galilean boost group-invariant solution of the system (3) is

$$u(t,x) = \frac{x+A}{\alpha t}, \;\; v(t,x) = \frac{C_2}{t} \tag{96}$$

where $A = -\alpha C_1$ and $t \neq 0$.

(iv) **The scaling**

$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} \tag{97}$$

By solving of the characteristic equations

$$\frac{\mathrm{d}t}{3t} = \frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}u}{2u} = -\frac{\mathrm{d}v}{2v}, \tag{98}$$

associated to this symmetry, we obtain the invariants

$$J_1 = \frac{x^3}{t}, \quad J_2 = ux^2, \quad J_3 = vx^2 \tag{99}$$

Generally, the group-invariant solution pair is

$$u(t,x) = \frac{f(\lambda)}{x^2}, \;\; v(t,x) = \frac{g(\lambda)}{x^2}, \quad \text{where} \quad \lambda = \frac{x^3}{t}, \tag{100}$$

and the functions f and g satisfy the system of third order nonlinear coupled ordinary DEs

$$2\alpha(g^2 - f^2) - \lambda^2 f' + 3\alpha\lambda(ff' - gg') + \beta(-24f + 24\lambda f' + 27\lambda^3 f''') = 0, \tag{101}$$

$$-4\alpha fg - \lambda^2 g' + 3\alpha\lambda(fg)' + \beta(-24g + 24\lambda g' + 27\lambda^3 g''') = 0 \tag{102}$$

(v) **Linear combination of time and space translations**

$$X = X_1 + cX_2. \tag{103}$$

We consider a symmetry X, which is a linear combination of the time and space translations symmetries, that is,

$$X = \frac{\partial}{\partial_t} + c\frac{\partial}{\partial x}, \tag{104}$$

for a constant c. The invariants associated to this symmetry X are

$$J_1 = x - ct, \quad J_2 = u, \quad J_3 = v \tag{105}$$

Hence, the invariant solution for the symmetry X is

$$u = f(x - ct), \quad v = g(x - ct), \tag{106}$$

for arbitrary functions f and g. Substitution of u and v from (106) into the system (3) yields a system of nonlinear third order ordinary DEs, namely

$$-cf'(\xi) + \alpha\{f(\xi)f'(\xi) - g(\xi)g'(\xi)\} + \beta f'''(\xi) = 0, \tag{107}$$

$$-cg'(\xi) + \alpha(f(\xi)g(\xi))' + \beta g'''(\xi) = 0, \tag{108}$$

which on integrating once with respect to ξ yields

$$-cf + \frac{1}{2}\alpha(f^2 - g^2) + \beta f'' + C_1 = 0, \tag{109}$$

for arbitrary constants C_1 and C_2.

$$-cg + \alpha fg + \beta g'' + C_2 = 0, \tag{110}$$

Remark 0.30. If we take the constants $C_1 = C_2 = 0$, then when the wave velocity $c = 0$, we can recover the stationary solutions given in (i).

Remark 0.31. Traveling wave solutions of the system (3) must satisfy the system (109).

Conclusion

In this paper, Lie group analysis was employed in studying a nonlinear coupled kdV system. A four-dimensional Lie algebra of symmetries was found for the nonlinear coupled system KdV equations. This was spanned by space and time translations, Galilean boost and scaling symmetries where the scaling symmetry acts on four variables. Associated to each symmetry, we obtained symmetry reductions that gave six nontrivial solutions for the coupled system. All the group-invariant solutions describe the various states of the system. The obtained solutions can be used as a benchmark against numerical simulations. In future, we will construct conservation laws use them to obtain exact solutions.

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Author's contribution

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