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On The Transcendental Equation $\sqrt[3]{y^2+x^2}=z^6$

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Abstract

The transcendental equation with three unknowns involving surd represented by the Diophantine equation $\sqrt[3]{x^2+y^2}=z^6$ is considered for its patterns of non-zero distinct integer solutions. A few properties between the solutions and special figurate numbers are presented.

Keywords: Transcendental equation, surd equation integral solutions

Notations

$$t_{3,s}=\frac{s(s+1)}{2} \qquad CT_{12,s}=6s(s+1)+1$$

$$P_s^3=\frac{s(s+1)(s+2)}{6} \qquad CT_{14,s}=7s(s+1)+1$$

$$P_s^4=\frac{s(s+1)(2s+1)}{6} \qquad CT_{16,s}=8s(s+1)+1$$

$$P_s^4=\frac{s(s+1)(2s+1)}{6} \qquad CT_{18,s}=9s(s+1)+1$$

$$P_s^5=\frac{s^2(s+1)}{2} \qquad CT_{20,s}=10s(s+1)+1$$

$$ST_s=6s(s-1)+1 \qquad ct_{4,s}=s^2+(s-1)^2$$

$$Pr_s=s(s+1)$$

Introduction

Diophantine equations have an unlimited field of research by reason of their variety. Most of the Diophantine problems are algebraic equations [1,2]. In [3-18], the integral solutions of transcendental equations involving surds are analysed for their respective integer solutions. This communication analyses a transcendental equation with three unknowns given by $\sqrt[3]{y^2+x^2}=z^6$. Infinitely many non-zero integer triples (x,y,z) satisfying the above equation are obtained. A few properties between the solutions and special figurate numbers are presented.

Method of analysis

The transcendental equation involving surds to be solved is

$$\sqrt[3]{y^2+x^2}=z^6 \tag{1}$$

The introduction of the transformations

$$x=m(m^2+n^2), y=n(m^2+n^2) \tag{2}$$

in (1) leads to

$$m^2 + n^2 = z^6 = (z^3)^2 \tag{3}$$

which is in the form of Pythagorean equation satisfied by

$$m = 2rs, n = r^2 - s^2, r \geq s \geq 0 \tag{4}$$

and

$$z^3 = r^2 + s^2 \tag{5}$$

Note that (5) is satisfied by

$$r = R(R^2 + S^2), s = S(R^2 + S^2) \tag{6}$$

and

$$z = (R^2 + S^2) \tag{7}$$

Substituting (6) in (4), one has

$$m = 2RS(R^2 + S^2)^2, n = (R^2 - S^2)(R^2 + S^2)^2, R \geq S \geq 0 \tag{8}$$

In view of (2), observe that

$$\left.\begin{array}{l} x = 2RS(R^2 + S^2)^8, \\ y = (R^2 - S^2)(R^2 + S^2)^8 \end{array}\right) \tag{9}$$

Thus, (7) and (9) represent the integer solutions to (1)
A few relations between the solutions and special figurate numbers are presented in Table 1 below.

Table 1: Relations

R	S	**Relation**
$S(S+1)$	S	$\frac{x}{z^8} = 4P_S^5$
$(S+1)$	S	$\frac{y^2}{z^{16}} = 8t_{3,S} + 1$
R	$R-1$	$\frac{3x}{z^8} = ST_R - 1$
$3S$	S	$\frac{x}{z^8} = 6S^2, a\ nastynumber$
$S^2 - 1$	S	$\frac{x}{z^8} = 12P_{S-1}^3$
$S+1$	S	$\frac{xy}{z^{16}} = 12P_S^4$
S^2	S	$3\left[4(t_{3,S})^2 - \frac{x+y}{z^8}\right] = 6S^2, a\ nastynumber$
$3(S+1)$	S	$\frac{x}{z^8} = CT_{12,S} - 1$
$4(S+1)$	S	$\frac{x}{z^8} = CT_{16,S} - 1$
$5(S+1)$	S	$\frac{x}{z^8} = CT_{20,S} - 1$
$7(S+1)$	S	$\frac{x}{z^8} = 2(CT_{14,S} - 1)$
$9(S+1)$	S	$\frac{x}{z^8} = 2(CT_{18,S} - 1)$
$3(S+1)$	S	$\frac{y}{z^8} - t_{18,S} \equiv 9(mod\ 25)$

Employing the integer solutions of (1), a few relations among the special polygonal and pyramidal numbers observed are presented below:

(i) $ct_{4,x+1} = 4t_{3,x} + 1$

(ii) $\left(\frac{2P_{x-1}^5}{t_{4,x-1}}\right)^2 + \left(\frac{P_y^5}{t_{3,y}}\right)^2 = \left(\frac{3P_z^3}{t_{3,z+1}}\right)^{18}$

(iii) $\left(\frac{12P_x^5}{ST_{x+1}-1}\right)^2 + \left(\frac{P_y^5}{t_{3,y}}\right)^2 = \left(\frac{6P_{z-2}^3}{Pr_{z-2}}\right)^{18}$

(iv) $\left(\frac{4P_x^5}{ct_{4,x+1}-1}\right)^2 + \left(\frac{P_y^5}{t_{3,y}}\right)^2 = \left(\frac{2P_{z-1}^5}{t_{4,z-1}}\right)^{18}$

Note 1:
Substituting (7) in (3) and employing the method of factorization, define

$$m + in = (R + iS)^6 \tag{10}$$

On equating the real and imaginary parts, it is seen that

$$m = R^6 - 15R^4S^2 + 15R^2S^4 - S^6 = f(R,S),$$

$$n = 6R^5S - 20R^3S^3 + 6RS^5 = g(R,S)$$

In view of (2), observe that

$$x = f(R,S)[(f(R,S))^2 + (g(R,S))^2],$$

$$y = g(R,S)[(f(R,S))^2 + (g(R,S))^2] \tag{11}$$

Thus, (7) and (11) represent the integer solutions to (1).

Note 2:
Apart from (2), the cube root on the L.H.S. of (1) may also be eliminated on choosing

$$x = (m^3 - 3mn^2), y = (3m^2n - n^3)$$

and the resulting equation is (3). In this case, the corresponding values of x, y are given by

$$x = f(R,S)[(f(R,S))^2 - 3(g(R,S))^2],$$

$$y = g(R,S)[3(f(R,S))^2 - (g(R,S))^2]$$

Note 3:
Write (3) as

$$m^2 + n^2 = z^6 * 1 \tag{12}$$

Assume 1 on the R.H.S. of (12) as

$$1 = \frac{(p^2-q^2+i2pq)(p^2-q^2-i2pq)}{(p^2+q^2)^2} \tag{13}$$

Substituting (7) and (13) in (12) and following the procedure as in Note:1, a different set of integer solutions to (1) is obtained.

Conclusion

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the transcendental equation with three unknowns involving surd represented by the equation in title. The researchers in this field may search for other choices of integer solutions to the considered surd equation.

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