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An application of Lie point symmetries in the study of a classical potential Burger's equation

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Abstract
A classical potential Burger's equation is studied by Lie group analysis. The constructed Lie point symmetries are used to perform symmetry reductions of the potential burgers equation and the resulting reduced ordinary differential equation systems investigated for exact group-invariant solutions. Solitons have also been constructed using symmetry span of space and time translations. Finally, two conserved quantities are derived by multiplier approach for the model.

Keywords: Symmetry Analysis, Group-Invariant Solutions, Stationary Solutions, Symmetry Reductions, Solitons, Potential Burgers Equation

1. Introduction
The Burgers equation [3] is of great mathematical and physical attention. The equation is ubiquitous in hydrodynamics applications and other scientific realms since it is among the simplest partial differential equations that combine a description of interacting nonlinear and dissipative effects. Mathematically, it does serve as the prototype of an equation that can be linearized through a direct coordinate transformation. Essentially, from the classical Burgers equation [2].

$$\Delta_0 \equiv q_t - 2qq_x - q_{xx} = 0, \tag{1.1}$$

we can let

$$q = u_x \tag{1.2}$$

Substitute for q in (1.1) and integrate with respect to x to yield the potential Burgers equation

$$\Delta \equiv u_t - u_x^2 - u_{xx} = 0 \tag{1.3}$$

where $u = u(t, x)$.

2. Preliminaries
In this section, we outline preliminary concepts which are useful in the sequel.

Local Lie groups
In the Euclidean spaces $\mathbb{R}^n$ of independent variables $x = x^i$ and $\mathbb{R}^m$ of dependent variables $u = u^\alpha$, we consider the transformations [13].

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2.1}$$

involving the continuous parameter ϵ which ranges from a neighbourhood $N' \subset N \subset \mathbb{R}$ of $\epsilon = 0$ where the functions ϕ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 2.1: The set $\mathcal{G}$ of transformations given by (2.1) is a local Lie group if it holds true that

1. (Closure) Given $T\epsilon_1, T\epsilon_2 \in \mathcal{G}$, for $\epsilon_1; \epsilon_2 \in N' \subset N$, then $T\epsilon_1 T\epsilon_2 = T\epsilon_3 \in G; \epsilon_3 = \phi(\epsilon_1; \epsilon_2) \in N$.
2. (Identity) There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T\epsilon T_0 = T_0 T\epsilon = T\epsilon$.
3. (Inverse) There exists a unique $T\epsilon\text{-}1 \in \mathcal{G}$ for every transformation $T\epsilon \in \mathcal{G}$, Where $\epsilon \in N' \subset N$ and $\epsilon^{-1} \in N$ such that $T\epsilon T\epsilon\text{-}1 = T\epsilon\text{-}1T = T_0$.

Remark 2.2: Associativity of the group $\mathcal{G}$ in (2.1) follows from (1).

Prolongations

In the system,

$$\Delta_\alpha\left(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{2.2}$$

the variables u^α are dependent. The partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \ldots, u_{(\pi)} = \{u_{i_1\ldots i_\pi}^\alpha\}$, are of the first, second, . . ., up to the πth-orders.

Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \tag{2.3}$$

the total differentiation operator with respect to the variables x^i and $\delta_j{}^i$, the Kronecker delta, we have

$$D_i(x^j) = \delta_i^j, ', \; u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(D_i(u^\alpha)), \quad \ldots, \tag{2.4}$$

where $u_i{}^\alpha$ defined in (2.4) are differential variables Ibragimov [8].

(i.) **Prolonged groups** Consider the local Lie group $\mathcal{G}$ given by the transformations

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{2.5}$$

where the symbol $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.3: The construction of the group $\mathcal{G}$ given by (2.5) is an equivalence of the computation of infinitesimal transformations

$$\begin{aligned} &\bar{x}^i \approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ &\bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{2.6}$$

obtained from (2.1) by a Taylor series expansion of $\phi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial\varphi^i(x^i, u^\alpha, \epsilon)}{\partial\epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial\psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial\epsilon}\Big|_{\epsilon=0} \tag{2.7}$$

Remark 2.4: The symbol of infinitesimal transformations, X, is used to write (2.6) as

$$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha, \tag{2.8}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.9}$$

is the generator of the group $\mathcal{G}$ given by (2.5).

Remark 2.5: To obtain transformed derivatives from (2.1), we use a change of variable formulae

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{2.10}$$

where $\bar{D}_j$ is the total differentiation in the variables $\bar{x}^i$. This means that

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \ \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{2.11}$$

If we apply the change of variable formula given in (2.10) on $\mathcal{G}$ given by (2.5), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \ \bar{D}_j(u^\alpha) = u_j^\alpha D_i(\varphi^j). \tag{2.12}$$

Expansion of (2.12) yields

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{2.13}$$

The variables $\bar{u}_i^\alpha$ can be written as functions of x^i, u^α, $u_{(1)}$, that is

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha. \tag{2.14}$$

Definition 2.6: The transformations in the space of the variables x^i, u^α, $u_{(1)}$ given in (2.5) and (2.14) form the first prolongation group $\mathcal{G}^{[1]}$.

Definition 2.7: Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon). \tag{2.15}$$

Remark 2.8: In terms of infinitesimal transformations, the first prolongation group $\mathcal{G}^{[1]}$ is given by (2.6) and (2.15).

ii) Prolonged generators

Definition 2.9: By using the relation given in (2.12) on the first prolongation group $\mathcal{G}^{[1]}$ given by Definition 2.6, we obtain [5]

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \quad \text{which gives} \tag{2.16}$$

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha \epsilon D_i \xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{2.17}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{2.18}$$

is the first prolongation formula.

Remark 2.10: Similarly, we get higher order prolongations [8],

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \ \ldots, \ \zeta_{i_1,\ldots,i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \tag{2.19}$$

Remark 2.11: The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \ldots, \mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \ \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1,\ldots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\ldots,i_\kappa}^\alpha}, \ \kappa \geq 1, \tag{2.20}$$

where X is the group generator given by (2.9).

Group invariants

Definition 2.12: A function $\Gamma(x^i, u^\alpha)$ is called an invariant of the group $\mathcal{G}$ of transformations given by (2.1) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \tag{2.21}$$

Theorem 2.13: *A function* $\Gamma(x^i, u^\alpha)$ *is an invariant of the group* $\mathcal{G}$ *given by (2.1) if and only if it solves the following first-order linear PDE:* [5]

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0. \tag{2.22}$$

From Theorem (2.13), we have the following result.

Theorem 2.14: *The local Lie group* $\mathcal{G}$ *of transformations in* $\mathbb{R}^n$ *given by (2.1)* [8] *has precisely* $n-1$ *functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \ldots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{2.23}$$

of the characteristic equations for (2.22):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \tag{2.24}$$

Symmetry groups

Definition 2.15: The vector field X (2.9) is a Lie point symmetry of the PDE system (2.2) if the determining equations

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \ldots, m, \quad \pi \geq 1, \tag{2.25}$$

are satisfied, where $\Big|_{\Delta_\alpha=0}$ means evaluated on $\Delta\alpha = 0$ and $X^{[\pi]}$ is the π-th prolongation of X.

Definition 2.16: The Lie group $\mathcal{G}$ is a symmetry group of the PDE system given in (2.2) if the PDE system (2.2) is form-invariant, that is

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \ldots, \bar{u}_{(\pi)}\right) = 0. \tag{2.26}$$

Theorem 2.17: *Given the infinitesimal transformations in (2.5), the Lie group* $\mathcal{G}$ *in (2.1) is found by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{2.27}$$

Lie algebras

Definition 2.18: A vector space V_r of operators [13] X (2.9) is a Lie algebra if for any two operators, X_i, $X_j \in V_r$, their commutator

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{2.28}$$

is in V_r for all $i, j = 1, \ldots, r$.

Remark 2.19: The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity [14].

Theorem 2.20: *The set of solutions of the determining equation given by (2.25) forms a Lie algebra* [5].

Conservation Laws

Let a system of πth-order PDEs be given by (2.2).

Definition 2.21: The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa\geq 1}(-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.29}$$

and the Lie- Backlund operator in abbreviated form [5] is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \ldots. \tag{2.30}$$

Remark 2.22: The Lie- Backlund operator (2.30) in its prolonged form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \ldots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{2.31}$$

Where

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \ldots, \zeta^\alpha_{i_1 \ldots i_\kappa} = D_{i_1 \ldots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_\kappa}, \ \ j = 1, \ldots, n. \tag{2.32}$$

and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{2.33}$$

Remark 2.23: The characteristic form of Lie- Backlund operator (2.31) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \ldots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}. \tag{2.34}$$

The method of multipliers

Definition 2.24: A function $\Lambda^\alpha\left(x^i, u^\alpha, u_{(1)}, \ldots\right) = \Lambda^\alpha,$ is a multiplier of the PDE system given by (2.2) if it satisfies the condition that [10]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{2.35}$$

where $D_i T^i$ is a divergence expression.

Definition 2.25: To find the multipliers Λ^α, one solves the determining equations (2.36) [1],

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha \Delta_\alpha) = 0. \tag{2.36}$$

Notation 2.26: We will use C_i, $i \in \mathbb{N}$ as constants of integration and $C_i(x_1, x_2, \ldots)$, $i \in \mathbb{N}$ as arbitrary function of $x_1, x_2, \ldots$.

3. Main results

3.1 Lie point symmetries of (1.3)

We start first by computing Lie point symmetries of the Burgers Equation (1.3)which admits the continuous Lie group of transformations infinitesimally generated by

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.1}$$

if and only if

$$X^{[2]}\Delta\Big|_{\Delta=0} = 0. \tag{3.2}$$

Using the definition of the second prolongation (2.20)

$$X^{[2]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{u_{tt}} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}$$

where

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\zeta_2 &= D_x(\eta) - u_x D_t(\tau) - u_x D_x(\xi), \\
\zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),
\end{aligned}$$

And

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_{tx}} + u_{tt} \frac{\partial}{\partial u_t} + \ldots \tag{3.3}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots \tag{3.4}$$

Now the determining Equation (3.2) yields

$$\zeta_1 - 2u_x \zeta_2 - \zeta_{22}|_{u_{xx}=u_t-u_x^2} = 0 \tag{3.5}$$

or

$$\begin{aligned}&\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u - 2u_x\eta_x - 2u_x^2\eta_u + 2u_tu_x\tau_x\\&+ 2u_tu_x^2\tau_u + 2u_x^2\xi_x + 2u_x^3\xi_u - \eta_{xx} - 2u_x\eta_{xu} - u_{xx}\eta_u - u_x^2\eta_{uu} + 2u_{xx}\xi_x\\&+ u_x\xi_{xx} + 2u_x^2\xi_{xu} + 3u_xu_{xx}\xi_u + u_x^3\xi_{uu} + 2u_{tx}\tau_x + u_t\tau_{xx} + 2u_tu_x\tau_{xu}\\&+ u_tu_{xx}\tau_u + 2u_xu_{tx}\tau_u + u_tu_x^2\tau_{uu}|_{u_{xx}=u_t-u_x^2} = 0\end{aligned} \tag{3.6}$$

which on substitution for u_{xx} by $u_t - u_x^2$ becomes

$$\begin{aligned}&\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u - 2u_x\eta_x - 2u_x^2\eta_u + 2u_tu_x\tau_x\\&+ 2u_tu_x^2\tau_u + 2u_x^2\xi_x + 2u_x^3\xi_u - \eta_{xx} - 2u_x\eta_{xu} - \{u_t - u_x^2\}\eta_u - u_x^2\eta_{uu}\\&+ 2\{u_t - u_x^2\}\xi_x + u_x\xi_{xx} + 2u_x^2\xi_{xu} + 3u_x\{u_t - u_x^2\}\xi_u + u_x^3\xi_{uu} + 2u_{tx}\tau_x\\&+ u_t\tau_{xx} + 2u_tu_x\tau_{xu} + u_t\{u_t - u_x^2\}\tau_u + 2u_xu_{tx}\tau_u + u_tu_x^2\tau_{uu} = 0\end{aligned} \tag{3.7}$$

By definition, τ, ξ and η are functions of t, x and u only. For that reason, we can then split Equation (3.7) on the derivatives of u (without losing any information) and obtain

$$\xi_u = \tau_u = \tau_x = 0, \tag{3.8}$$

$$\eta_{uu} + \eta_u = 0, \tag{3.9}$$

$$\xi_{xx} - 2\eta_{xu} - 2\eta_x - \xi_t = 0, \tag{3.10}$$

$$2\xi_x - \tau_t = 0, \tag{3.11}$$

$$\eta_t - \eta_{xx} = 0. \tag{3.12}$$

By the Equations in (3.8),

$$\tau = \tau(t), \quad \text{and} \quad \xi = \xi(t, x). \tag{3.13}$$

Equation (3.9) is second order ordinary linear differential equation and is solved by

$$\eta = \Omega(t, x) + \Gamma(t, x)e^{-u}. \tag{3.14}$$

Substitute for ξ and η in (3.10) to obtain

$$\xi_{xx} + \Gamma_x(t, x)e^{-u} - 2\Omega_x(t, x) - 2\Gamma_x(t, x)e^{-u} - \xi_t = 0. \tag{3.15}$$

Furthermore, Equation (3.11) is necessary and sufficient for

$$\xi = \frac{\tau_t}{2}x + a(t), \tag{3.16}$$

and when used in (3.15), one obtains

$$\Omega_x = -\frac{\tau_{tt}}{4}x - \frac{a_t(t)}{2}, \tag{3.17}$$

from which

$$\Omega = -\frac{\tau_{tt}}{8}x^2 - \frac{a_t(t)}{2}x + b(t). \tag{3.18}$$

If we use the current values for η, τ and ξ in (3.12), we have that

$$-\frac{\tau_{ttt}}{8}x^2 - \frac{a_{tt}(t)}{2}x + b_t(t) + \Gamma_t(t,x)e^{-u} + \frac{\tau_{tt}}{4} - \Gamma_{xx}(t,x)e^{-u} = 0 \tag{3.19}$$

which splits on powers of x to yield

$$x^2 : \frac{\tau_{ttt}}{8} = 0, \tag{3.20}$$

$$x : \frac{a_{tt}(t)}{2} = 0, \tag{3.21}$$

$$x^0 : \{\Gamma_t(t,x) - \Gamma_{xx}(t,x)\}e^{-u} + b_t(t) + \frac{\tau_{tt}}{4} = 0. \tag{3.22}$$

Equations (3.20) and (3.21) respectively admit

$$\tau = 4c_1t^2 + 8c_2t + c_3 \tag{3.23}$$

$$a(t) = 2c_4t + c_5 \tag{3.24}$$

Splitting Equation (3.22) on coefficients of e^{-u}, and using value of τ in the resulting equations, one finds that

$$b(t) = -2c_1t + c_6 \tag{3.25}$$

Hence we have

$$\begin{aligned}
\tau &= 4c_1t^2 + 8c_2t + c_3, \\
\xi &= 4c_1tx + 4c_2x + 2c_4t + c_5, \\
\eta &= -c_1(x^2 + 2t) - c_4x + c_6 + \Gamma(t,x)e^{-u},
\end{aligned}$$

where $\Gamma(t, x)$ is any solution to the heat equation

$$\Gamma_t(t,x) - \Gamma_{xx}(t,x) = 0. \tag{3.26}$$

We have obtained an infinite dimensional Lie algebra spanned by

$$X_1 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 2t)\frac{\partial}{\partial u} \tag{3.27}$$

$$X_2 = 8t\frac{\partial}{\partial t} + 4x\frac{\partial}{\partial x} \tag{3.28}$$

$$X_3 = \frac{\partial}{\partial t} \tag{3.29}$$

$$X_4 = 2t\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} \tag{3.30}$$

$$X_5 = \frac{\partial}{\partial x} \tag{3.31}$$

$$X_6 = \frac{\partial}{\partial u} \tag{3.32}$$

$$X_\Gamma = \Gamma(t,x)e^{-u}\frac{\partial}{\partial u}. \tag{3.33}$$

Remark 3.1: The potential Burgers equation (1.3) has an infinite-dimensional Lie algebra of point symmetries and many higher symmetries [4]. This is evident from the presence of an arbitrary function of the independent variables in the last symmetry.

3.2 Commutator Table for Symmetries

We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket [14], for example, we have that

$$[X_5, X_3] = X_5X_3 - X_3X_5 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{3.34}$$

Remark 3.2: The remaining commutation relations are obtained analogously. We present all commutation relations in table (1) below.

Table 1: A commutator table for the Lie algebra spanned by the symmetries of pontential Burger's equation

$[X_i; X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X
X_1	0	-8 X_1	$-X_1 + 2X_6$	0	$-2X_4$	0	1
X_2	8 X_1	0	$-8X_3$	$4X_4$	$-4X_5$	0	2
X_3	X_2 $2X_6$	$8X_3$	0	$2X_5$	0	0	t
X_4	0	$-4X_4$	$-2X_5$	0	X_6	0	3
X_5	$2X_4$	$4X_5$	$-X_6$	0	0	0	x
X_6	0	0	0	0	0	0	-X
X	- 1	- 2	- t	- 3	- x	X	0

Where

$$\begin{aligned}
\Gamma_1 &= 4t^2\Gamma_t + 4tx\Gamma_x + (x^2+2t)\Gamma \\
\Gamma_2 &= 8t\Gamma_t + 4x\Gamma_x \\
\Gamma_3 &= 2t\Gamma_x + x\Gamma
\end{aligned}$$

3.3 Group Transformations

The corresponding one-parameter group of transformations can be determined by solving the Lie equations [15]. Let $T_{\epsilon i}$ be the group of transformations for each X_i, i = 1, 2, 3, 4. We display how to obtain $T_{\epsilon i}$ from X_i by finding one-parameter group for the infinitesimal generator X_5, namely,

$$X_5 = \frac{\partial}{\partial x}. \tag{3.35}$$

In particular, we have the Lie equations

$$\begin{aligned}
\frac{\mathrm{d}\bar{t}}{\mathrm{d}\epsilon} &= 0, \quad \bar{t}\Big|_{\epsilon=0} = t, \\
\frac{\mathrm{d}\bar{x}}{\mathrm{d}\epsilon} &= 1, \quad \bar{x}\Big|_{\epsilon=0} = x, \\
\frac{\mathrm{d}\bar{u}}{\mathrm{d}\epsilon} &= 0, \quad \bar{u}\Big|_{\epsilon=0} = u.
\end{aligned} \tag{3.36}$$

Solving the system (3.36) one obtains,

$$\bar{t} = t, \quad \bar{x} = x + \epsilon, \quad \bar{u} = u, \tag{3.37}$$

and hence the one-parameter group $T_{\epsilon 4}$ corresponding to the operator X_5 is

$$T_{\epsilon_5}: \quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_5, u) \tag{3.38}$$

All the five one-parameter groups are presented below:

$$\begin{aligned}
T_{\epsilon_1}: &\quad (\bar{t}, \bar{x}, \bar{u}) = \left(\frac{t}{1-4\epsilon_1 t}, xe^{4\epsilon_1 t}, u - (x^2+2t)\epsilon_1\right) \\
T_{\epsilon_2}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (te^{8\epsilon_2}, xe^{4\epsilon_2}, u) \\
T_{\epsilon_3}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t + \epsilon_3, x, u) \\
T_{\epsilon_4}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + 2\epsilon_4 t, u - \epsilon_4 x). \\
T_{\epsilon_5}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_5, u). \\
T_{\epsilon_6}: &\quad (\bar{t}, \bar{x}, \bar{u}) = (t, x, u + \epsilon_6). \\
T_{\epsilon_\Gamma}: &\quad (\bar{t}, \bar{x}, \bar{u}) = \big(t, x, \ln|\Gamma(t,x)\epsilon_\Gamma + e^u|\big).
\end{aligned} \tag{3.39}$$

3.4 Symmetry transformations

We now show how the symmetries we have obtained can be used to transform sepcial exact solutions of the potential burgers equation into new solutions. The Lie group analysis vouches for fundamental ways of constructing exact solutions of PDEs, that is, group transformations of known solutions and construction of group-invariant solutions. We will illustrate these methods with examples. If $\bar{u} = g(\bar{t}, \bar{x})$ is a solution of equation (1.3)

$$\phi(t, x, u, \epsilon) = g(f_1(t, x, u, \epsilon), f_2(t, x, u, \epsilon)), \tag{3.40}$$

is also a solution. The one parameter groups dictate to the following generated solutions:

$$\begin{aligned} T_{\epsilon_1} : u =& g\Big(\frac{t}{1-4\epsilon_1 t}, xe^{4\epsilon_1 t}\Big) + (x^2+2t)\epsilon_1 \\ T_{\epsilon_2} : u =& g(te^{8\epsilon_2}, xe^{4\epsilon_2}) \\ T_{\epsilon_3} : u =& g(t+\epsilon_3, x) \\ T_{\epsilon_4} : u =& g(t, x+2\epsilon_4 t) + \epsilon_4 x. \\ T_{\epsilon_5} : u =& g(t, x+\epsilon_5). \\ T_{\epsilon_6} : u =& g(t, x) - \epsilon_6. \\ T_\Gamma : u =& \ln\left|e^{g(t,x)} - \Gamma(t,x)_{\epsilon_\Gamma}\right| \end{aligned} \tag{3.41}$$

3.5 Construction of Group-Invariant Solutions

Now we compute the group invariant solutions of Burger's equation.

1) $X_1 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2+2t)\frac{\partial}{\partial u}$

The associated Lagrangian equations

$$\frac{\mathrm{d}t}{4t^2} = \frac{\mathrm{d}x}{4tx} = \frac{\mathrm{d}u}{-(x^2+2t)}, \tag{3.42}$$

yield two invariants, $J_1 = x/t$ and $J_2 = u + 1/2 \ln x + x^2/4t$. Thus using $J_2 = \Phi(J_1)$, we have

$$u(t, x) = \Phi\big(\frac{x}{t}\big) - \frac{x^2}{4t} - \frac{1}{2}\ln x. \tag{3.43}$$

The derivatives are given by:

$$\begin{aligned} u_t =& -\frac{x}{t^2}\Phi' big\big(\frac{x}{t}\big) + \frac{x^2}{4t^2} \\ u_x =& \frac{1}{t}\Phi'\big(\frac{x}{t}\big) - \frac{x}{2t} - \frac{1}{2x}, \\ u_{xx} =& \frac{1}{t^2}\Phi''\big(\frac{x}{t}\big) - \frac{1}{2t} + \frac{1}{2x^2}. \end{aligned}$$

If we substitute these derivatives into Equation (1.3), we obtain the second order ordinary differential equation

$$\frac{1}{t^2}\Phi''\big(\frac{x}{t}\big) + \frac{1}{t^2}\Phi'^2\big(\frac{x}{t}\big) - \frac{1}{tx}\Phi'\big(\frac{x}{t}\big) + \frac{3}{4x^2} = 0.$$

By the transformation

$$z = \frac{x}{t}, \quad \text{and} \quad y = \Phi'(z), \tag{3.44}$$

we have the Riccati equation

$$z^2y' + z^2y^2 - zy + \frac{3}{4} = 0, \tag{3.45}$$

for which one particular solution is

$$y_1 = \frac{1}{2z}.$$

Suppose we let

$$y = y_1 + \frac{1}{v},$$

and obtain

$$\frac{\mathrm{d}y}{\mathrm{d}z} = -\frac{1}{2z^2} - \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}z}.$$

Substitutions for y and $\mathrm{d}y/\mathrm{d}z$ into Equation (3.45), gives us

$$-\frac{1}{2z^2} - \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}z} = -\left(\frac{1}{2z} + \frac{1}{v}\right)^2 + \frac{1}{z}\left(\frac{1}{2z} + \frac{1}{v}\right) - \frac{3}{4z^2}, \tag{3.46}$$

that simplifies to

$$\frac{\mathrm{d}v}{\mathrm{d}z} = 1 \tag{3.47}$$

Integration yields

$$v = z + C_1 \tag{3.48}$$

Finally

$$y = \frac{1}{2z} + \frac{1}{z + C_1},$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{1}{2z} + \frac{1}{z + C_1},$$

whose integration yields

$$\Phi(z) = \frac{1}{2}\ln z + \ln|z + C_1| + C_2 \tag{3.49}$$

Thus, the group-invariant solution associated to the X_1 is

$$u(t,x) = \ln\left|\frac{1}{\sqrt{t}}\left(\frac{x}{t} + C_1\right)\right| - \frac{x^2}{4t} + C_2$$

ii) $X_2 = 8t\frac{\partial}{\partial t} + 4x\frac{\partial}{\partial x}$

$$\frac{\mathrm{d}t}{8t} = \frac{\mathrm{d}x}{4x} = \frac{\mathrm{d}u}{0} \tag{3.50}$$

This gives the constants $J_1 = u$ and $J_2 = \frac{x}{\sqrt{t}}$, giving the solution

$$u = \phi\left(\frac{x}{\sqrt{t}}\right) \tag{3.51}$$

We obtain the derivatives as follows:

$$u_t = -\frac{1}{2}\frac{x}{(\sqrt{t})^3}\phi'\left(\frac{x}{\sqrt{t}}\right), \tag{3.52}$$

$$u_x = \frac{1}{\sqrt{t}}\phi'\left(\frac{x}{\sqrt{t}}\right) \tag{3.53}$$

$$u_{xx} = \frac{1}{t}\phi''\left(\frac{x}{\sqrt{t}}\right) \tag{3.54}$$

If we substitute the above derivatives in Equation (1.3), we obtain the second order ordinary differential equation

$$-\frac{1}{2}\frac{x}{(\sqrt{t})^3}\phi'\left(\frac{x}{\sqrt{t}}\right)-\frac{1}{t}\phi'^2\left(\frac{x}{\sqrt{t}}\right)-\frac{1}{t}\phi''\left(\frac{x}{\sqrt{t}}\right)=0 \tag{3.56}$$

The transformations $y=\frac{x}{\sqrt{t}}$ and $Q=\phi'(y)$,, yields the Bernoulli equation (with n = 2),

$$2\frac{\mathrm{d}Q}{\mathrm{d}y}+yQ=-2Q^2 \tag{3.57}$$

Equation (3.57) admits

$$Q=\left(e^{\frac{y^2}{4}}\{\sqrt{\pi}\,\mathrm{erf}((\frac{y}{2}))+C_1\}\right)^{-1} \tag{3.58}$$

This means that

$$\frac{\mathrm{d}\phi}{\mathrm{d}y}=\left(e^{\frac{y^2}{4}}\{\sqrt{\pi}\,\mathrm{erf}((\frac{y}{2}))+C_1\}\right)^{-1}, \tag{3.59}$$

and by integration with respect to *y*, we have

$$\phi(y)=\ln\left(\left|\sqrt{\pi}\,\mathrm{erf}((\frac{y}{2}))+C_1\right|\right)+C_2 \tag{3.60}$$

Finally, by the change of variables to the initial ones, we have

$$u(t,x)=\ln\left(\left|\sqrt{\pi}\,\mathrm{erf}((\frac{y}{2}))+C_1\right|\right)+C_2 \tag{3.61}$$

(iii) $X_3 = \partial/\partial t$ (**Stationary solutions**)

The Lagrangian system associated with the operator X_3 is

$$\frac{\mathrm{d}t}{8t}=\frac{\mathrm{d}x}{4x}=\frac{\mathrm{d}u}{0}, \tag{3.62}$$

whose invariants are $J_1 = x$ and $J_2 = u$. So, $u = \psi(x)$ is the group-invariant solution. Substituting of $u = \psi(x)$ into (1.3) yields

$$\psi''(x)+\psi'^2(x)=0. \tag{3.63}$$

Equation (3.63) is a second order nonlinear ODE which is satisfied by the function

$$\psi(x)=\ln|x-C_1|+C_2. \tag{3.64}$$

Thus the stationary solution for (1.3) is given by

$$u(t,x)=\ln|x-C_1|+C_2. \tag{3.65}$$

iv $X_4=2t\frac{\partial}{\partial x}-\frac{\partial}{\partial u}$

Characteristic equations associated to the operator X_4 are

$$\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}x}{2t}=\frac{\mathrm{d}u}{-x}, \tag{3.66}$$

yields $J1 = t$ and $J_2 = x^2/2 + 2tu$. As a result, the group-invariant solution of (1.3) for this case is $J_2 = \varphi(J_1)$, for φ an arbitrary function. That is,

$$u(t,x)=\delta(t)-\frac{x^2}{4t},\quad \delta(t)=\frac{\phi(t)}{2t},\quad t\neq 0 \tag{3.67}$$

Substitution of the value of *u* from equation (3.67) into equation (1.3) yields a first order ordinary differential equation $\delta'(t)+\frac{1}{2t}=0$, whose general solution is $\delta(t)=-\frac{1}{2}\ln|t|+C_4$. Hence, the group-invariant solution under X_4 is

$$u(t,x) = -\Big(\frac{1}{2}\ln|t| + \frac{x^2}{4t}\Big) + C_4, \quad t \neq 0 \tag{3.68}$$

v) **Space translation -invariant solutions**
We consider the space translation operator

$$X_5 = \frac{\partial}{\partial x} \tag{3.69}$$

Characteristic equations associated with the operator (3.69) are

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}, \tag{3.70}$$

which give two invariants $J_1 = t$ and $J_2 = u$. Therefore, $u = \psi(t)$ is the group-invariant solution for some arbitrary function ψ. Substitution of $u = \psi(t)$ into (1.3) yields

$$\psi'(t) = 0, \tag{3.71}$$

whose solution is

$$\psi(t) = C_1, \tag{3.72}$$

for C_1 an arbitrary constant. Hence the group-invariant solution of (1.3) under the space translation operator (3.69) is

$$u(t,x) = C_1 \tag{3.73}$$

(vi) $X_6 = \frac{\partial}{\partial u}$
This Lie point symmetry does not have any invariant solution.

vii) X_Γ
This Lie point symmetry does not have any invariant solution.

3.6 Soliton
We obtain a traveling wave solution of the potential Burgers Equation (1.3) by considering a linear combination of the symmetries X_5 and X_3, namely, [13]

$$X = cX_5 + X_3 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \tag{3.74}$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} \tag{3.75}$$

We get two invariants, $J_1 = x - ct$ and $J_2 = u$. So the group-invariant solution is

$$u(t,x) = \Phi(x - ct), \tag{3.76}$$

for some arbitrary function Φ and c the velocity of the wave.
Substitution of u into (1.3) yields a second order ordinary differential equation

$$c\Phi' + \Phi'^2 + \Phi'' = 0, \tag{3.77}$$

with constant coefficients. If $z = x - ct$ and $\Phi'(z) = y$, then we have a simplified ordinary differential equation of the form

$$y' + y^2 + cy = 0, \tag{3.78}$$

whose solution is

$$y(z) = \left(-\frac{1}{k} + C_1 e^{kz}\right)^{-1} \tag{3.79}$$

This means that

$$\Phi'(z) = \left(-\frac{1}{k} + C_1 e^{kz} \right)^{-1}, \tag{3.80}$$

and is solved by

$$\Phi(z) = \ln\left(\left| C_1 k e^{kz} - 1 \right| \right) - kz + C_2. \tag{3.81}$$

Clearly,

$$u(t,x) = \ln\left(\left| C_1 k e^{cx - c^2 t} - 1 \right| \right) - cx - cx^2 t + C_2 \tag{3.82}$$

4 Conservation laws of equation (1.3)

We will employ multipliers in the construction of conservation laws.

4.1 The multipliers

We make use of the Euler-Lagrange operator defined as defined in [15] to look for a zeroth order multiplier $\Lambda = \Lambda(t, x, u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t - u_x^2 - u_{xx}\}\right] = 0. \tag{4.1}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + \dots \tag{4.2}$$

Expansion of Equation (4.1) yields

$$\Lambda_u(u_t - u_x^2 - u_{xx}) - D_t(\Lambda) + 2D_x(\Lambda u_x) - D_x^2(\Lambda) = 0 \tag{4.3}$$

Invoking the total derivatives

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{4.4}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots. \tag{4.5}$$

on Equation (4.3) produces

$$2(\Lambda_x - \Lambda_{xu})u_x + 2(\Lambda - \Lambda_u)u_{xx} + (\Lambda_u - \Lambda_{uu})u_x^2 - \Lambda_t - \Lambda_{xx} = 0 \tag{4.6}$$

Splitting Equation (4.6) on derivatives of u produces an overdetermined system of four partial differential equations, namely

$$u_x : \Lambda_x - \Lambda_{xu} = 0, \tag{4.7}$$

$$u_{xx} : \Lambda - \Lambda_u = 0, \tag{4.8}$$

$$u_x^2 : \Lambda_u - \Lambda_{uu} = 0, \tag{4.9}$$

$$\text{rest} : \Lambda_t + \Lambda_{xx} = 0 \tag{4.10}$$

Note that Equation (4.8) is sufficient for Equations (4.9) and (4.7) . We can write Equation (4.8) as

$$\Lambda = \frac{\mathrm{d}\Lambda}{\mathrm{d}u} \tag{4.11}$$

if and only if

$$\frac{d\Lambda}{\Lambda} = du, \tag{4.12}$$

giving a solution of the form

$$\Lambda = A(t, x)e^u \tag{4.13}$$

Substitute this value into Equation (4.10) to obtain

$$(A_t + A_{xx})e^u = 0 \tag{4.14}$$

which is the same as

$$A_t + A_{xx} = 0. \tag{4.15}$$

This is a linear heat equation and can be solved by separation of variables. If we assume a solution of the form

$$A(t, x) = X(x)T(t), \tag{4.16}$$

then Equation (4.15) gives

$$X''(x)T(t) + X(x)T'(t) = 0. \tag{4.17}$$

Dividing by $XT \neq 0$ and introducing the separation constant $-\lambda^2$, we have

$$T' - \lambda^2 T = 0 \tag{4.18}$$

$$X'' + \lambda^2 X = 0 \tag{4.19}$$

The solutions to Equations (4.18) and (4.19) are respectively given by

$$T(t) = C_1 e^{\lambda^2 t} \tag{4.20}$$

$$X(x) = C_2 \cos \lambda x + C_3 \sin \lambda x \tag{4.21}$$

which implies that

$$A(t, x) = e^{\lambda^2 t}[\mathcal{C}_1 \cos \lambda x + \mathcal{C}_2 \sin \lambda x], \quad \mathcal{C}_1 = C_1C_2, \ \ \mathcal{C}_2 = C_1C_3. \tag{4.22}$$

We finally have Equation (4.13) becomes

$$\Lambda(t, x) = e^{\lambda^2 t + u}[\mathcal{C}_1 \cos \lambda x + \mathcal{C}_2 \sin \lambda x] \tag{4.23}$$

Essentially, we extract the two multiplies

$$\Lambda_1 = e^{\lambda^2 t + u} \cos \lambda x \tag{4.24}$$

$$\Lambda_2 = e^{\lambda^2 t + u} \sin \lambda x. \tag{4.25}$$

Remark 4.1: Recall that a multiplier Λ for Equation(1.3) has the property that for the density $T^t = T^t(t, x, u)$ and flux $T^x = T^x(t, x, u, u_x)$

$$\Lambda\left(u_t - u_x^2 - u_{xx}\right) = D_t T^t + D_x T^x \tag{4.26}$$

We derive a conservation law corresponding to each of the multipliers.

(i). **Conservation law for the multiplier** $\Lambda_1 = e^{\lambda^2 t + u} \cos \lambda x$
Expansion of equation (4.26) gives

$$e^{\lambda^2 t + u} \cos \lambda x\{u_t - u_x^2 - u_{xx}\} = T_t^t + u_t T_u^t + + T_x^x + u_x T_u^x + u_{xx} T_{u_x}^x \tag{4.27}$$

Splitting Equation (4.27) on the second derivative of u yields

$$u_{xx} \;:\; T^x_{u_x} = -e^{\lambda^2 t+u}\cos\lambda x, \tag{4.28}$$

$$\text{Rest} \;:\; e^{\lambda^2 t+u}\cos\lambda x\{u_t - u_x^2\} = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x. \tag{4.29}$$

The integration of Equation (4.28) with respect to ux gives

$$T^x = \{-e^{\lambda^2 t+u}\cos\lambda x\}u_x + A(t,x,u) \tag{4.30}$$

Substituting the expression of T^x from (4.30) into Equation (4.29) we get

$$e^{\lambda^2 t+u}\cos\lambda x\{u_t - u_x^2\} = T^t_t + T^t_u u_t + \lambda u_x e^{\lambda^2 t+u}\sin\lambda x + A_x + u_x^2\{-e^{\lambda^2 t+u}\cos\lambda x\} + A_u u_x. \tag{4.31}$$

which splits on first derivatives of u, to give

$$u_x \;:\; A_u = -\lambda e^{\lambda^2 t+u}\sin\lambda x, \tag{4.33}$$

$$u_t \;:\; T^t_u = e^{\lambda^2 t+u}\cos\lambda x, \tag{4.34}$$

$$\text{Rest} \;:\; 0 = T^t_t + A_x. \tag{4.35}$$

Integrating equations (4.33) and (4.34) with respect to u manifests that

$$T^t = e^{\lambda^2 t+u}\cos\lambda x + C(t,x), \tag{4.36}$$

$$A = -\lambda e^{\lambda^2 t+u}\sin\lambda x + B(t,x) \tag{4.37}$$

By substituting the obtained functions into Equation (4.35), we have

$$C_t(t,x) + B_x(t,x) = 0 \tag{4.38}$$

Since $C(t, x)$ and $B(t, x)$ contribute to the trivial part of the conservation law, we take $C(t, x) = B(t, x) = 0$ and obtain the conserved quantities

$$T^t = e^{\lambda^2 t+u}\cos\lambda x, \tag{4.39}$$

$$T^x = -e^{\lambda^2 t+u}\{u_x\cos\lambda x + \lambda\sin\lambda x\} \tag{4.40}$$

from which the conservation law corresponding to the multiplier $\Lambda_1 = e^{\lambda^2 t+u}\cos\lambda x$ is given by

$$D_t\left(e^{\lambda^2 t+u}\cos\lambda x\right) - D_x\left(e^{\lambda^2 t+u}\{u_x\cos\lambda x + \lambda\sin\lambda x\}\right) = 0. \tag{4.41}$$

(ii). **Conservation law for the multiplier** $\Lambda_2 = e^{\lambda^2 t+u}\sin\lambda x$
Expansion of equation (4.26) gives

$$e^{\lambda^2 t+u}\sin\lambda x\{u_t - u_x^2 - u_{xx}\} = T^t_t + u_t T^t_u + +T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} \tag{4.42}$$

Splitting Equation (4.42) on the second derivative of u yields

$$u_{xx} \;:\; T^x_{u_x} = -e^{\lambda^2 t+u}\sin\lambda x, \tag{4.43}$$

$$\text{Rest} \;:\; e^{\lambda^2 t+u}\sin\lambda x\{u_t - u_x^2\} = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x. \tag{4.44}$$

The integration of Equation (4.43) with respect to ux gives

$$T^x = \{-e^{\lambda^2 t+u}\sin\lambda x\}u_x + a(t,x,u). \tag{4.45}$$

Substituting the expression of T x from (4.45) into Equation (4.44) we get

$$e^{\lambda^2 t+u}\sin\lambda x\{u_t-u_x^2\} = T_t^t + T_u^t u_t - \lambda u_x e^{\lambda^2 t+u}\cos\lambda x + a_x + u_x^2\{-e^{\lambda^2 t+u}\sin\lambda x\} + a_u u_x. \quad (4.46)$$

which splits on first derivatives of u, to give

$$u_x \quad : \quad a_u = \lambda e^{\lambda^2 t+u}\cos\lambda x, \quad (4.48)$$

$$u_t \quad : \quad T_u^t = e^{\lambda^2 t+u}\sin\lambda x, \quad (4.49)$$

$$\text{Rest} \quad : \quad 0 = T_t^t + a_x. \quad (4.50)$$

Integrating equations (4.48) and (4.49) with respect to u manifests that

$$T^t = e^{\lambda^2 t+u}\sin\lambda x + c(t,x), \quad (4.51)$$

$$a = \lambda e^{\lambda^2 t+u}\cos\lambda x + b(t,x). \quad (4.52)$$

By substituting the obtained functions into Equation (4.50), we have

$$c_t(t,x) + b_x(t,x) = 0 \quad (4.53)$$

We may take $c(t, x)$ and $c(t, x)$ as contributing to the trivial part of the conservation law and set them to $c(t, x) = b(t, x) = 0$ and obtain the conserved quantities

$$T^t = e^{\lambda^2 t+u}\sin\lambda x, \quad (4.54)$$

$$T^x = -\,e^{\lambda^2 t+u}\{u_x\sin\lambda x - \lambda\cos\lambda x\} \quad (4.55)$$

from which the conservation law corresponding to the multiplier $\Lambda_2 = e^{\lambda^2 t+u}\sin\lambda x$ is given by

$$D_t\left(e^{\lambda^2 t+u}\sin\lambda x\right) - D_x\left(e^{\lambda^2 t+u}\{u_x\sin\lambda x - \lambda\cos\lambda x\}\right) = 0. \quad (4.56)$$

Remark 4.2: It can be shown that the two sets of conserved quantities are conservation laws. Given that $e^{\lambda^2 t+u} \neq 0$and $\sin\lambda x \neq 0$, the verification reaffirms that the potential burger's equation is itself a conversation law.

5. Conclusion

In this manuscript, an infinite dimensional Lie algebra of Lie point symmetries has been applied to study a potential Burger's equation. A commutator table has been constructed for the obtained Lie algebra. We have also used symmetry reductions to compute exact group-invariant solutions, including a soliton. Conservation laws have also been derived for the model with the use of zeroth order multipliers.

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7. Author's contribution

The author wrote the article as a scholarly duty and passion to disseminate mathematical research and hereby declares that there is no conflict of interest.

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